**Regularity of the free boundary for the two-phase Bernoulli problem**

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## Summary of the main results

Given a bounded open set $D \subset \mathbb{R}^d$, $d \geq 2$, we consider the two-phase functional

$$J_{\text{TP}}(u, D) := \int_D |\nabla u|^2\,dx + \lambda_+^2 \big|\Omega_u^+ \cap D\big| + \lambda_-^2 \big|\Omega_u^- \cap D\big|, \tag{TP}$$

where $\lambda_+$ and $\lambda_-$ are non-negative constants and where we set $\Omega_u^+ = \{u > 0\}$ and $\Omega_u^- = \{u < 0\}$. The paper is dedicated to the regularity of the free boundary $\partial\Omega_u^+ \cup \partial\Omega_u^-$ of local minimizers $u$ of $J_{\text{TP}}$ in $D$.

**History.** The functional $J_{\text{TP}}$ was introduced in [ACF], where the authors studied the regularity of the free boundary in $d = 2$, in the case

$$\lambda_+ > 0 \qquad \text{and} \qquad \lambda_- = 0, \tag{Case 1}$$

and showed that $\partial\Omega_u^+ = \partial\Omega_u^-$ in $D$ and is a $C^1$-regular curve.
In $d > 2$, the regularity, still under the condition (Case 1), was first obtained by Caffarelli in the seminal papers [Caf1, Caf2]; more recently, a new viscosity solution approach was developed in [DFS1, DFS2].

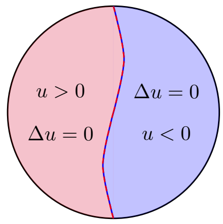


Fig.1. $\partial\Omega_u^\pm$ in (Case 1).

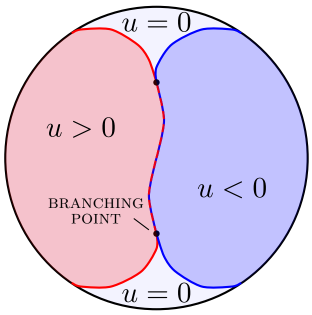


Fig.2. $\partial\Omega_u^\pm$ in (Case 2).

**Main result.** The behavior of $\partial\Omega_u^+$ and $\partial\Omega_u^-$ changes completely when

$$\lambda_+ > 0 \qquad \text{and} \qquad \lambda_- > 0, \tag{Case 2}$$

as the two free boundaries may not coincide and branching points may appear (see Fig.2). In fact, the first regularity result in this case was obtained only recently. First, in dimension $d = 2$, in [SV], we proved that the free boundaries $\partial\Omega_u^+$ and $\partial\Omega_u^-$ are $C^{1,\alpha}$-regular curves, while in [STV] we obtained the same result for almost minimizers.

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| In this paper, we prove that, in any dimension $d \geq 2$, $\partial\Omega_u^+$ and $\partial\Omega_u^-$ are $C^{1,\alpha}$-regular manifolds in a neighborhood of $\partial\Omega_u^+ \cap \partial\Omega_u^-$. |

In particular, this completes the analysis started in [ACF] and is a first step towards the description of the singularities of the vectorial Bernoulli problem (see [MTV2]) introduced in [CSY, KL, MTV].

**Applications.** As a consequence of our anaysis, we obtain (in any dimension $d \geq 2$) a $C^{1,\alpha}$-regularity result for the optimal sets, solutions of the shape optimization problem (introduced in [BuV] and [BoV])

$$\min\Big\{\sum_{i=1}^n \big(\lambda_1(\Omega_i) + m_i|\Omega_i|\big) \;:\; \Omega_i \subset D \text{ open}\,;\; \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j\Big\}.$$

where $m_i > 0$ and $\lambda_1(\Omega_i)$ denotes the first eigenvalue for the Dirichlet Laplacian on $\Omega_i$. On the right, a numerical simulation (where $D$ is the thorus and $n = 8$) by Bogosel (see [BoV]). The case $m_i = 0$, for all $i$, is the classical optimal partition problems, for which the regularity is well-known (see [CL1, CL2, CTV] and the references therein).

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