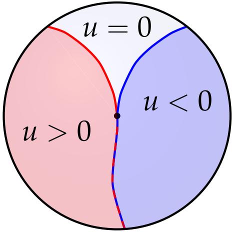
On the regularity of the two-phase free boundaries



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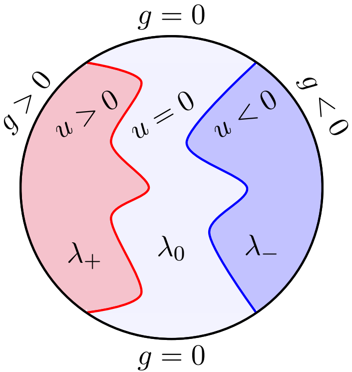
Given:

– *a domain* $D \subseteq \mathbb{R}^d$ *(we assume* $D = B_1$*),*

– *positive constants* λ_+, λ_- *, and* λ_0*,*

– *a boundary datum* $g : \partial D \to \mathbb{R}$*,*



minimize *the* TWO-PHASE *functional*

$$J_{\mathrm{TP}}(u, D) = \int_D |\nabla u|^2 \, dx + \lambda_+^2 \big|\{u > 0\} \cap D\big| + \lambda_-^2 \big|\{u < 0\} \cap D\big| + \lambda_0^2 \big|\{u = 0\} \cap D\big|,$$

among *all functions* $u : D \to \mathbb{R}$ **such that** $u = g$ *on* ∂D.

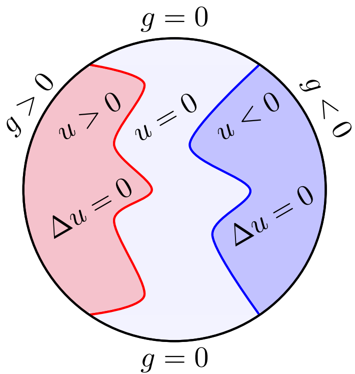
First considerations:

1. *The solutions u are* **harmonic** *in the set* $\{u \neq 0\}$.

2. *Is there an* **equation for** u *in the entire* $D = B_1$?

$$< \Delta u, \varphi >:= \int_D \nabla u \cdot \nabla \varphi$$

$$= \int_{\{u \neq 0\}} \varphi \, \Delta u + \int_{\partial\{u\neq 0\}} \frac{\partial u}{\partial n} \varphi = \int_{\partial\{u\neq 0\}} \frac{\partial u}{\partial n} \varphi$$



Then

$$\Delta u = |\nabla u_+| \left(\mathcal{H}^{d-1} \llcorner \partial\{u > 0\}\right) - |\nabla u_-| \left(\mathcal{H}^{d-1} \llcorner \partial\{u < 0\}\right) \quad \text{in} \quad D$$

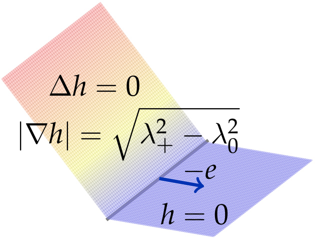
$\boxed{\text{Solution } u}$ $\leftrightarrows$ $\boxed{\text{Free boundary } \partial\{u > 0\} \cup \partial\{u < 0\}}$

1981 Alt-Caffarelli (*J. Reine Agnew. Math.*) $-$ $u \in C^{0,1}_{loc}(D)$ $-$ *for* $u \geq 0$,
1984 Alt-Caffarelli-Friedman (*Trans. Amer. Math. Soc.*) $-$ $u \in C^{0,1}_{loc}(D)$ $-$ *for any* u.

The Lipschitz continuity of u is optimal.
In fact, for every $\lambda_+ > \lambda_0 = 0$ and $e \in \partial B_1$,
the function

$$h(x) = \left(\lambda_+^2 - \lambda_0^2\right)^{1/2} \max\{0, x \cdot e\}$$

is a local minimizer in $\mathbb{R}^d$.



Corollary: • $\Omega_u^+ = \{u > 0\}$ and $\Omega_u^- = \{u < 0\}$ are open sets ;
• $\Delta u = 0$ in Ω_u^+ and $\Delta u = 0$ in Ω_u^- .

Question: What is the regularity of the free boundary $\partial\Omega_u^+ \cup \partial\Omega_u^- \cap D$?

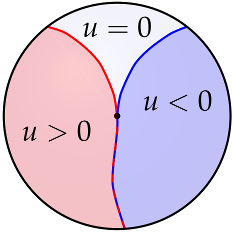
On the regularity of the two-phase free boundaries

PART I

Known results,

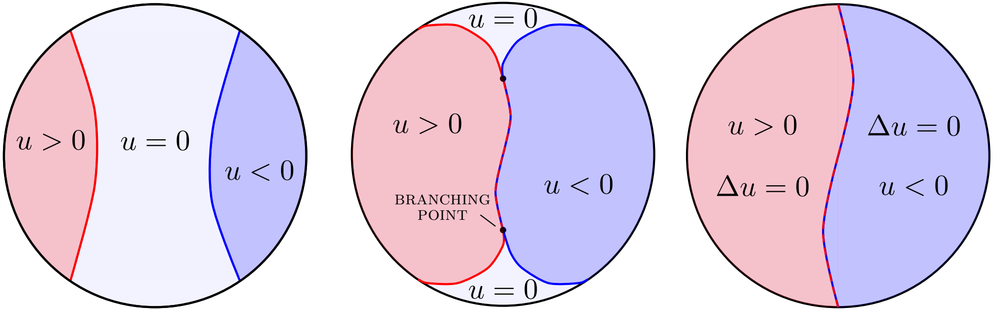
main theorem,

and applications



Suppose that $\operatorname{dist}\left(\{g>0\},\{g<0\}\right)>0$ *on the sphere.*

...and consider the following three cases.



$\lambda_+ \gg \lambda_0$ *and* $\lambda_- \gg \lambda_0$

$\lambda_+ > \lambda_0$ *and* $\lambda_- > \lambda_0$

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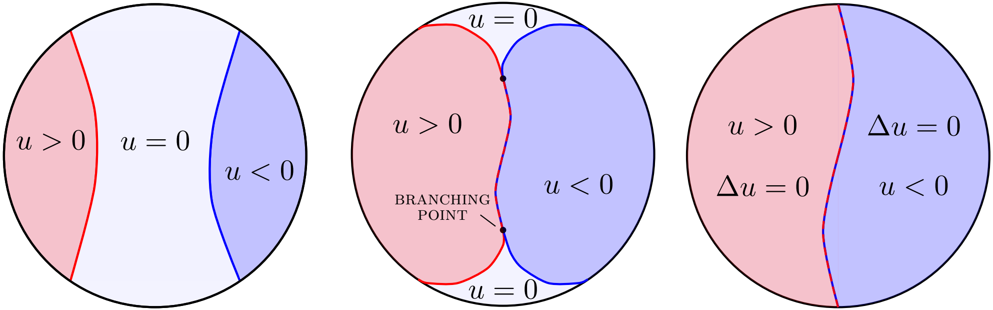

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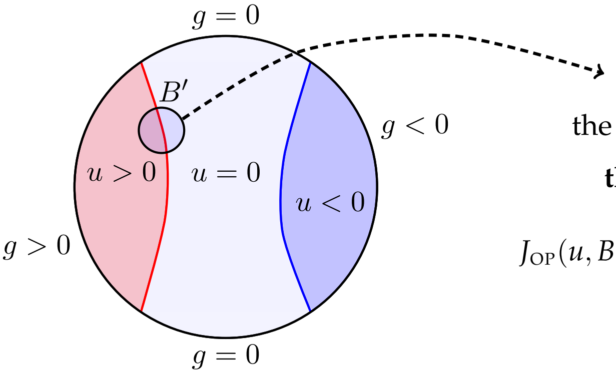
Recall that $\operatorname{dist}\left(\{g>0\},\{g<0\}\right)>0$ *on the sphere.*

Suppose that $\lambda_+ \gg \lambda_0$ and $\lambda_- \gg \lambda_0$.

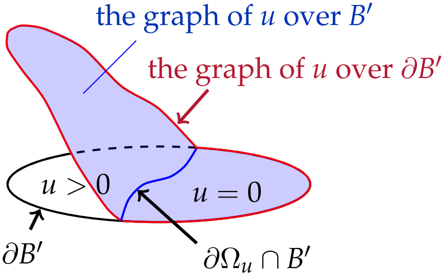


In the small ball B'

the positive part u_+ minimizes

the one-phase functional

$$J_{\mathrm{OP}}(u,B_1)=\int_{B_1}|\nabla u|^2\,dx$$
$$+\left(\lambda_+^2-\lambda_0^2\right)\left|\{u>0\}\cap B_1\right|$$



Definition: We say that u is *a local minimizer of* J_{OP} *in* B' if

$$J_{\mathrm{OP}}(u, B') \leq J_{\mathrm{OP}}(v, B')$$

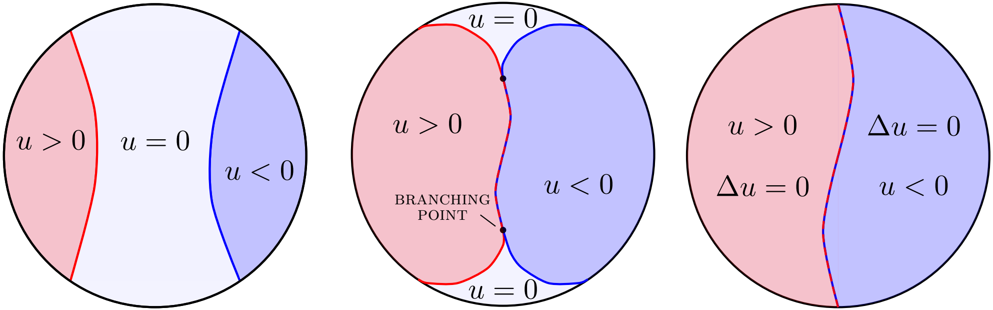
for every $v : B' \to \mathbb{R}$ such that $u = v$ on $\partial B'$.

Theorem *(Alt-Caffarelli'81, Weiss'00)***.** There is $d^* \in \{5, 6, 7\}$ such that:
If u is a (nonnegative) local minimizer of J_{OP} in $B' \subseteq \mathbb{R}^d$,
then the free boundary decomposes as: $\partial\Omega_u^+ \cap B' = \mathrm{Reg}\,(\partial\Omega_u^+) \cup \mathrm{Sing}\,(\partial\Omega_u^+)$

- $\mathrm{Reg}\,(\partial\Omega_u^+)$ is a $C^{1,\alpha}$-regular manifold;
- $\mathrm{Sing}\,(\partial\Omega_u^+)$ is empty ($d < d^*$), discrete ($d = d^*$), of dimension $d - d^*$ ($d > d^*$).

Suppose that $\operatorname{dist}\left(\{g>0\},\{g<0\}\right)>0$ *on the sphere.*

...and consider the following three cases.



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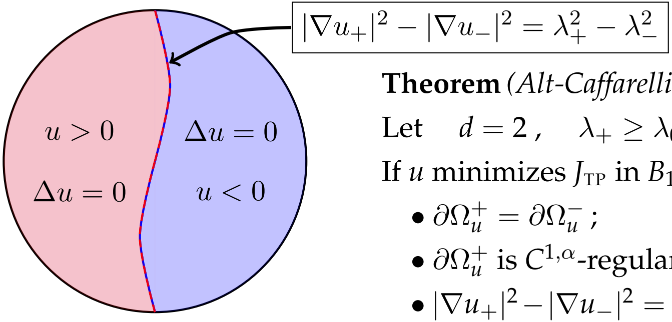

Theorem *(Alt-Caffarelli-Friedman'84)*.

Let $d = 2$, $\lambda_+ \geq \lambda_0$ and $\lambda_- = \lambda_0$.

If u minimizes J_{TP} in B_1, then:

- $\partial\Omega_u^+ = \partial\Omega_u^-$;
- $\partial\Omega_u^+$ is $C^{1,\alpha}$-regular curve ;
- $|\nabla u_+|^2 - |\nabla u_-|^2 = \lambda_+^2 - \lambda_-^2$ on $\partial\Omega_u^+ \cap B_1$.

Regularity of free boundaries satisfying a transmission condition:

1987 - 1989 Caffarelli (*Comm. Pure Appl. Math.,...*) — Harnack inequality approach;
2005 Caffarelli-Salsa — *A geometric approach to free boundary problems* (book);
2014-2018 De Silva-Ferrari-Salsa — Partial Boundary Harnack approach
(**2010** De Silva — Partial Boundary Harnack for the one-phase pb).

Suppose that $\mathrm{dist}\big(\{g>0\},\{g<0\}\big)>0$ *on the sphere.*

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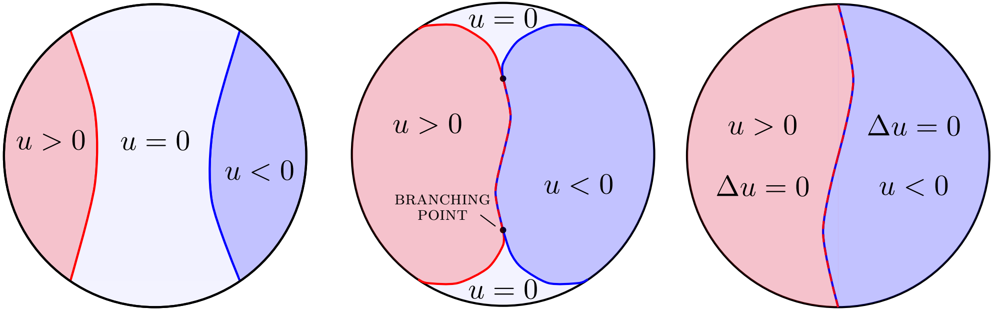

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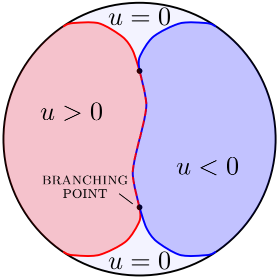


Assume $\lambda_+ > \lambda_0$ and $\lambda_- > \lambda_0$.

W.l.o.g. $\lambda_0 = 0$.

Decomposition of the free boundary:

- **one-phase points:**

  $\Gamma^+_{\mathrm{OP}} = \partial\Omega^+_u \setminus \partial\Omega^-_u$ and $\Gamma^-_{\mathrm{OP}} = \partial\Omega^-_u \setminus \partial\Omega^+_u$
- **two-phase points:** $\Gamma_{\mathrm{TP}} = \partial\Omega^+_u \cap \partial\Omega^-_u$

Theorem. Let $d \geq 2$. Let u be a minimizer of J_{TP} with $\lambda_0 = 0$, $\lambda_+ > 0$ and $\lambda_- > 0$.
Then, in a neighborhood of any $x_0 \in \partial\Omega^+_u \cap \partial\Omega^-_u \cap D$,
the free boundaries $\partial\Omega^+_u$ and $\partial\Omega^-_u$ are $C^{1,\alpha}$-regular manifolds.

2017 Spolaor-Velichkov (*Comm. Pure Appl. Math.*) – the case $d = 2$;
2018 Spolaor-Trey-Velichkov (*Comm. PDE*) – almost-minimizers in $\mathbb{R}^2$;
2019 De Philippis-Spolaor-Velichkov (*to appear*) – any $d \geq 2$.

Corollary (Alt-Caffarelli; Weiss; Spolaor-Trey-V.; De Philippis-Spolaor-V.)**.**
Let $d \geq 2$ and $D \subseteq \mathbb{R}^d$ be an open set.
Let u be a local minimizer of J_{TP} in D with $\lambda_0 = 0$, $\lambda_+ > 0$ and $\lambda_- > 0$.
Then, for each of the sets Ω_u^+ and Ω_u^-, the free boundary $\partial\Omega_u^\pm \cap D$

can be decomposed as $\partial\Omega_u^\pm \cap D = \text{Reg}\,(\partial\Omega_u^\pm) \cup \text{Sing}\,(\partial\Omega_u^\pm)$, where:

- the regular part $\text{Reg}\,(\partial\Omega_u^\pm)$ is a $C^{1,\alpha}$ manifold;
- $\text{Sing}\,(\partial\Omega_u^\pm)$ is a (possibly empty) closed set of **one-phase singularities**, and
  - $\text{Sing}\,(\partial\Omega_u^\pm)$ is empty, if $d < d^*$;
  - $\text{Sing}\,(\partial\Omega_u^\pm)$ is discrete, if $d = d^*$;
  - $\text{Sing}\,(\partial\Omega_u^\pm)$ has Hausdorff dimension $d - d^*$, if $d > d^*$;
  - $\text{Sing}\,(\partial\Omega_u^\pm) \cap \partial\Omega_u^+ \cap \partial\Omega_u^- = \emptyset$.

Minimize $\sum_{k=1}^{n} \Big(\lambda_1(\Omega_k) + m_k|\Omega_k|\Big)$ among

all n-uples of sets $(\Omega_1, \ldots, \Omega_n)$ such that:

- $\Omega_k \subseteq D$, where D is a $C^{1,\alpha}$-regular box;
- $\Omega_k \cap \Omega_j = \emptyset$, whenever $k \neq j$.

**Numerical simulations by Beniamin Bogosel*
(`http://www.cmap.polytechnique.fr/ beniamin.bogosel/`)

Theorem. Let $d = 2$. Let $(\Omega_1,..., \Omega_n)$ be a solution of the multiphase problem in a $C^{1,\alpha}$-regular box D. Then each of the sets Ω_k is $C^{1,\alpha}$-regular.

2015 Bogosel-Velichkov (*SIAM Numer. Anal.*)
2019 Spolaor-Trey-Velichkov (*Comm. PDE*)

Theorem (regularity in higher dimension - part I). Let $d \geq 2$. Let $(\Omega_1, \ldots, \Omega_n)$ be a solution of the multiphase problem in the $C^{1,\alpha}$-regular box $D \subseteq \mathbb{R}^d$. Then:

- each of the sets Ω_k is open and has finite perimeter

  2014 Bucur-Velichkov (*SIAM Contr. Optim.*) ;
- there are no triple points: $\partial\Omega_i \cap \partial\Omega_j \cap \partial\Omega_k = \emptyset$

  2014 Bucur-Velichkov,

  2014 Velichkov – three-phase monotonicity formula,

  2015 Bogosel-Velichkov (*SIAM Numer. Anal.*) - new proof in 2D;
- there are no two-phase points on the boundary of the box: $\partial\Omega_i \cap \partial\Omega_j \cap \partial D = \emptyset$

  2015 Bogosel-Velichkov;

Theorem (regularity in higher dimension - part I). Let $d \geq 2$. Let $(\Omega_1, \dots, \Omega_n)$ be a solution of the multiphase problem in the $C^{1,\alpha}$-regular box $D \subseteq \mathbb{R}^d$. Then:

Theorem (regularity in higher dimension - part II).

- the one-phase parts of the free boundaries $\partial\Omega_k$ are as regular as the solutions of the one-phase (Alt-Caffarelli) problem:

  2009 Briançon-Lamboley (*Ann. IHP*) – **1981** Alt-Caffarelli (*Crelle*).

  Regularity of the sets Ω *that minimize* $\lambda_1(\Omega)$ *with measure constraint* $|\Omega| = c$.

- in a neighbourhood of $\partial D \cap \partial\Omega_k$ the free boundary $\partial\Omega_k$ is a $C^{1,\alpha}$-manifold:

  2019 Russ-Trey-Velichkov – **2018** Chang Lara-Savin;

  Regularity for minimizers of $\lambda_1(\Omega)$ *under the constraints* $|\Omega| = c$ *and* $\Omega \subseteq D$.

- in a neighbourhood of $\partial\Omega_j \cap \partial\Omega_k$ the free boundary $\partial\Omega_k$ is a $C^{1,\alpha}$-manifold:

  2019 De Philippis-Spolaor-Velichkov.

Minimize $\sum_{k=1}^{n} \Big(\lambda_1(\Omega_k) + m_k|\Omega_k|\Big)$ among all n-uples of sets $(\Omega_1, \dots, \Omega_n)$ such that:

- $\Omega_k \subseteq D$, where D is a $C^{1,\alpha}$-regular box;
- $\Omega_k \cap \Omega_j = \emptyset$, whenever $k \neq j$.

Theorem. Let $d \geq 2$. Let $(\Omega_1,\dots,\Omega_n)$ be a solution of the multiphase problem in a $C^{1,\alpha}$-regular box D. Then, for each of the sets Ω_k, the free boundary can be decomposed as $\partial\Omega_k = \mathrm{Reg}\,(\partial\Omega_k) \cup \mathrm{Sing}\,(\partial\Omega_k)$, where:

$\mathrm{Reg}\,(\partial\Omega_k)$ is a $C^{1,\alpha}$ manifold,

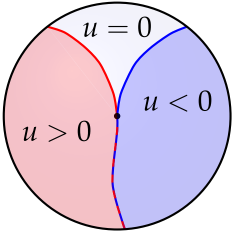
- $\mathrm{Sing}\,(\partial\Omega_k)$ is a closed set of **one-phase singularities**, such that:
  - $\mathrm{Sing}\,(\partial\Omega_k)$ is empty, if $d < d^*$, and discrete, if $d = d^*$;
  - $\mathrm{Sing}\,(\partial\Omega_k)$ has Hausdorff dimension $d - d^*$, if $d > d^*$;
  - $\mathrm{Sing}\,(\partial\Omega_k)$ lies on a positive distance from ∂D and $\partial\Omega_j$, for $j \neq k$.

On the regularity of the two-phase free boundaries

PART II

About the proof,

of the main theorem



Theorem. Let $d \geq 2$. Let $u : B_1 \to \mathbb{R}$ be a local minimizer (in B_1) of the functional

$$J_{\text{TP}}(u, B_1) = \int_{B_1} |\nabla u|^2 \, dx + \lambda_+^2 \, \big|\{u > 0\} \cap B_1\big| + \lambda_-^2 \, \big|\{u < 0\} \cap B_1\big|.$$

with $\lambda_+ > 0$ and $\lambda_- > 0$.

Then, in a neighborhood of any two-phase point $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap D$,

the free boundaries $\partial\Omega_u^+$ and $\partial\Omega_u^-$ are $C^{1,\alpha}$-regular manifolds.

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2018 Spolaor-Trey-Velichkov (*Comm. PDE*) – almost-minimizers in $\mathbb{R}^2$;
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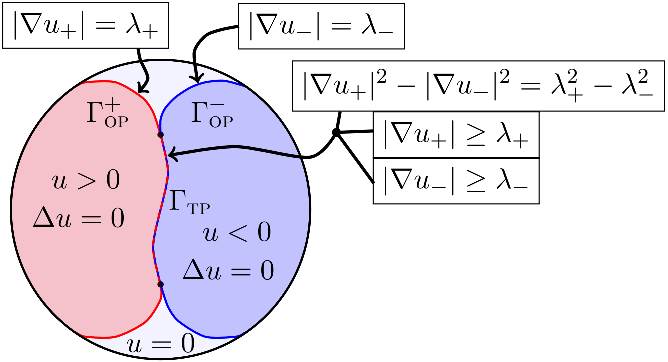
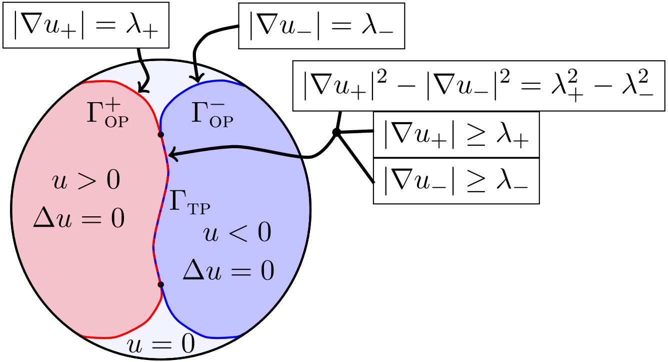
Heuristics.



Decomposition of u and J_{TP}. $J_{\mathrm{TP}}(u, B_1) = J_{\mathrm{OP}}^{\lambda_+}(u_+) + J_{\mathrm{OP}}^{\lambda_-}(u_-)$, where

$$J_{\mathrm{OP}}^{\lambda}(\varphi) = \int_{B_1} |\nabla \varphi|^2\, dx + \lambda^2 \left|\{\varphi > 0\} \cap B_1\right|.$$

Heuristics.

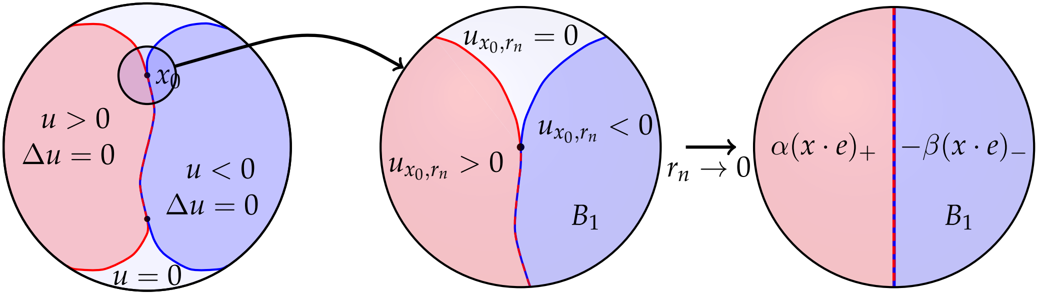


Variation of J_{OP}. Let ξ be a smooth vector field. Then

$$\delta J_{\mathrm{OP}}^{\lambda_+}(u_+)[\xi] = \frac{d}{dt}\Big|_{t=0} J_{\mathrm{OP}}^{\lambda_+}\Big(u_+\big(x - t\xi(x)\big)\Big) = \int_{\partial\{u>0\}} \Big(-|\nabla u_+|^2 + \lambda_+^2\Big)(\xi \cdot n).$$

Let $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^+ \cap B_1$. Let $r_n \to 0$.

Let $u_{x_0,r_n}(x) := \frac{1}{r_n} u(x_0 + r_n x)$.



u_{x_0,r_n} converges uniformly to $\alpha(x \cdot e)_+ - \beta(x \cdot e)_-$
$$\begin{cases} \alpha^2 - \beta^2 = \lambda_+^2 - \lambda_-^2 \\ \alpha \geq \lambda_+ \quad \text{and} \quad \beta \geq \lambda_- \end{cases}$$

Startegy of the proof *(in any dimension)*. Show that:

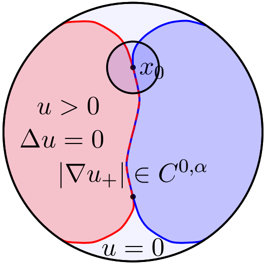
(a) Differentiability. ∇u_+ exists at every $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^+$.

(b) OP-TP transition. $|\nabla u_+|$ is Hölder continuous on $\partial\Omega_u^+$.

(c) Flatness. u_+ is flat around every $x_0 = 0 \in \partial\Omega_u^+ \cap \partial\Omega_u^+$:

$$\alpha(x \cdot e - \varepsilon r)_+ \leq u_+(x) \leq \alpha(x \cdot e + \varepsilon r)_+ \quad \text{in} \quad B_r.$$

for some $e \in \partial B_1$, $r > 0$ and $\varepsilon > 0$.



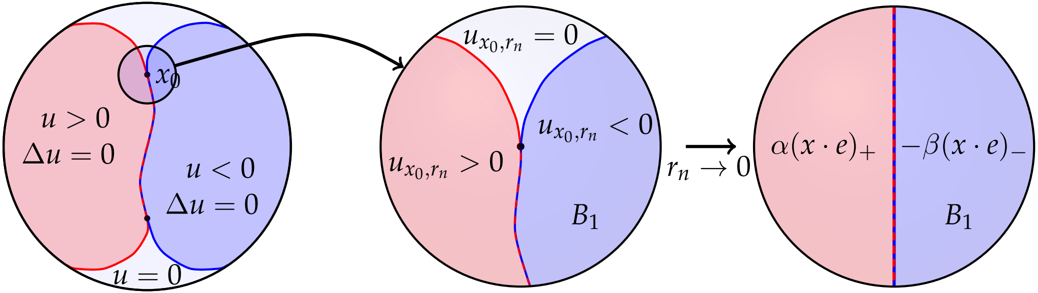
Apply the following Theorem (De Silva. *Interf. Free Bound.* 2010).

If **(a)** , **(b)** *and* **(c)** *do hold, then* $\partial\Omega_u^+$ *is* $C^{1,\alpha}$ *regular in* $B_{r/2}$.

REMARK: **Differentiability** $\Leftrightarrow$ UNIQUENESS OF THE BLOW-UP LIMIT

Let $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^+ \cap B_1$. Let $r_n \to 0$.

Let $u_{x_0,r_n}(x) := \frac{1}{r} u(x_0 + r_n x)$.



u_{x_0,r_n} converges uniformly to $\alpha(x \cdot e)_+ - \beta(x \cdot e)_-$

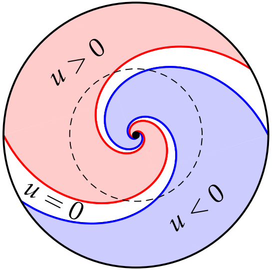
$$\begin{cases} \alpha^2 - \beta^2 = \lambda_+^2 - \lambda_-^2 \\ \alpha \geq \lambda_+ \quad \text{and} \quad \beta \geq \lambda_- \end{cases}$$

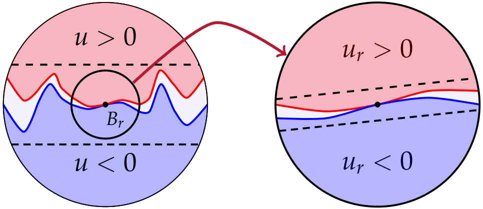
Remark. The blow-up limit (a priori) depends on the blow-up sequence.

This happens at the tip of a spiral:

Spirals => no differentiability!

We have to exclude this spiraling behavior!



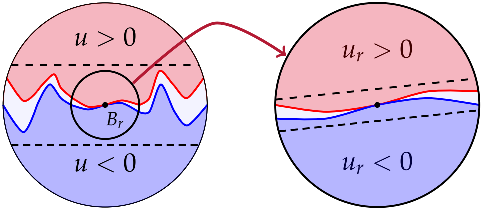


Lemma (*Improvement of flatness*).

If $\|u - H_{\alpha,e}\|_{L^\infty(B_1)} \le \varepsilon_0$, then

$$\|u_r - H_{\alpha',e'}\|_{L^\infty} \le (1-\delta)\|u - H_{\alpha,e}\|_{L^\infty}$$

where $|e-e'|$ and $|\alpha-\alpha'|$ are small.



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where $|e-e'|$ and $|\alpha-\alpha'|$ are small.

Corollary 1 *(Uniqueness of the blow-up limit)*.

If $\|u - H_{\alpha,e}\|_{L^\infty(B_1)} \leq \varepsilon_0$, then there are constants $\gamma > 0$ and $C > 0$ such that

$$\|u_r - H_{\alpha,e}\|_{L^\infty(B_1)} \leq Cr^\gamma \quad \text{for every} \quad r \in (0,1).$$

Corollary 2 *(Differentiability)*. Fixed $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$,

$$u_+(x) = \alpha\, e \cdot (x - x_0) + C|x - x_0|^{1+\gamma} \quad \text{for every} \quad x \in \Omega_u^+.$$

Take any $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$ and $y_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$.

Then $\|u_{r,x_0} - H_{\alpha,e}\| \leq Cr^\gamma$ and $\|u_{r,y_0} - H_{\alpha',e'}\| \leq Cr^\gamma$ for all $r > 0$.

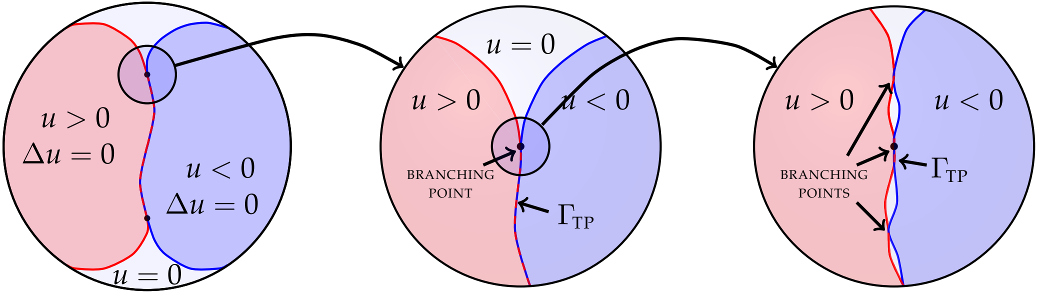
$$\|\cdot\| = \|\cdot\|_{L^\infty(B_1)}$$

$$\begin{aligned}
\big|\nabla u_+(x_0) - \nabla u_+(y_0)\big| &= \big|\alpha\, e - \alpha'\, e'\big| \\
&\approx \|H_{\alpha,e} - H_{\alpha',e'}\| \\
&\leq \|u_{r,x_0} - H_{\alpha,e}\| + \|u_{r,y_0} - H_{\alpha',e'}\| + \|u_{r,x_0} - u_{r,y_0}\| \\
&\leq Cr^\gamma + Cr^\gamma + \mathrm{Lip}(u)\,\tfrac{|x_0-y_0|}{r}.
\end{aligned}$$

Choose r such that $|x_0 - y_0| = r^{1+\gamma}$. Then

$$\big|\nabla u_+(x_0) - \nabla u_+(y_0)\big| \lesssim |x_0 - y_0|^{\frac{\gamma}{1+\gamma}}.$$

- $|\nabla u| = \lambda_+$ on the **one-phase** free boundary $\partial\Omega_u^+ \setminus \partial\Omega_u^-$;
- $|\nabla u|$ is Hölder continuous on the **two-phase** free boundary $\partial\Omega_u^+ \cap \partial\Omega_u^-$.

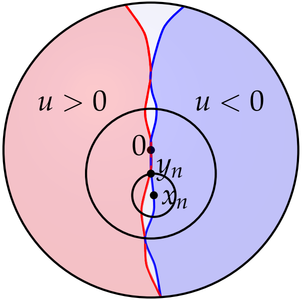


Branching point: $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$ such that

$$\big|B_r(x_0) \cap \{u = 0\}\big| > 0 \quad \text{for every} \quad r > 0.$$

Let x_n be a sequence of **one-phase points** converging to the **branching point** $0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$.

Aim: Prove that $|\nabla u_+(0)| = \lambda_+$.



$|\nabla u_+(0) - \nabla u_+(y_n)| \lesssim |y_n|^\gamma$

and so, $|\alpha(0) - \alpha(y_n)| \lesssim |y_n|^\gamma$

$\left|u(x) - H_{\alpha(y_n)}(x)\right| \lesssim r_n^{1+\gamma}$ in $B_{4r_n}(y_n)$,

where $r_n = |x_n - y_n|$.

But then $\left|u_+(x) - H^+_{\lambda_+}(x)\right| = o(r_n)$ in $B_{r_n}(x_n)$

and so, $\alpha(0) = \lambda_+$. $\square$