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Stable minimal cones with an isolated singularity

Tesi di Laurea

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'Happy or sad?'
'Sad'
'Okay. But I warn you. I'll break your heart.'
'Already broken.'

Abstract

In the present thesis, we study the regularity of minimizers of two functionals.

The first one is the perimeter. A set of locally finite perimeter E in $\mathbb{R}^{n+1}$ is, roughly speaking, a set that supports a divergence theorem, where the boundary measure, that is called perimeter measure, is the n-dimensional Hausdorff measure restricted to a thiner part of ∂E: the reduced boundary $\partial^* E$.

The second one is an energy whose stationary points model the equilibrium state of incompressible fluids in absence of gravity: $\mathcal{A}_\theta(E) := \mathcal{H}^n(\partial^* E \cap \mathbb{R}^{n+1}_+) - \cos(\theta)\mathcal{H}^n(\partial^* E \cap \mathbb{R}^n)$, where $\mathbb{R}^{n+1}_+$ is the upper half-space with boundary $\mathbb{R}^n$, E is a set of locally finite perimeter in $\mathbb{R}^{n+1}_+$, and θ is the contact angle between $\partial^* E \cap \mathbb{R}^{n+1}_+$ and $\mathbb{R}^n$ (whenever E is a stationary point for $\mathcal{A}_\theta$).

In both cases it was proved that, if E is a minimizer, $\overline{\partial E \cap H}$ is a smooth hypersurface away from a closed set $\Sigma(E)$, called the singular set of E. Here $H = \mathbb{R}^{n+1}$ when E minimizes the perimeter, and $H = \mathbb{R}^{n+1}_+$ when E minimizes $\mathcal{A}$. In both cases the Federer's dimension reduction principle allows to relate the Hausdorff dimension of the singular set to the analysis of minimizing cones that are smooth away from the origin. Thus, in this thesis we focus on the study of cones, and we prove the Simons' Theorem, obtaining as a consequence that $\Sigma(E)$ is empty whenever E is a perimeter minimizer and $n \leq 6$.

Then, we follow the work of Chodosh, Edelen and Li in [2], proving, for a minimizer E of $\mathcal{A}_\theta$, that $\Sigma(E)$ is empty when $n \leq 3$, when $n = 4$ and θ is small enough, and when $n \leq 6$ and θ is close enough to $\frac{\pi}{2}$.

Moreover, we give a new proof of the case $n = 3$, based on the analogous work of Jerison and Savin (see [3]) about the regularity of minimizers of the Alt-Caffarelli functional. Assuming the additional hypothesis that E is the region below the graph of a Lipschitz function, we think that would be possible to apply the same arguments also in the case $n = 4$ for small contact angles.

Contents

Summary

In the first Chapter, we start by introducing the theory of sets of locally finite perimeter, recalling their basic properties, and the definition of perimeter minimizer. We then prove the first and second variation formulae for the perimeter. We obtain that, for a smooth perimeter minimizer E with boundary ∂E, ∂E has zero mean curvature, and E satisfies a stability inequality (1.7). The stability inequality carries the information of A, i.e. the tangential gradient of the external unit normal to E. We briefly explain that the Federer's dimension reduction argument, together with the regularity of the reduced bounbdary $\partial^* E$ of a perimeter minimizer E, allow us to focus on the study of the non-existence of open cones minimizing the perimeter with an isolated singularity. In this way, we can estimate the Hausdorff dimension of the singular set of a minimizer. We then prove the Simons' inequality (1.3.9) for cones with an isolated singularity, and with zero mean curvature at the boundary. We finally prove the Simons' Theorem(1.3.10), about the non existence in $\mathbb{R}^n$ ($2 \leq n \leq 7$), of smooth cones that minimize the perimeter, and with an isolated singularity. The strategy of this proof is to plug a competitor depending on $|A|$ in the stability inequality, and then to use the Simons' inequality in order to prove that $|A|$ must be zero, and thus the cone must have been an half-space.

In the second Chapter we introduce the capillary functional (2.1) $\mathcal{A}^\theta$, defined for sets of locally finite perimeter in $\mathbb{R}^{n+1}_+$. Here θ will be the fixed contact angle that smooth minimizers of $\mathcal{A}^\theta$ form with $\mathbb{R}^n := \partial\mathbb{R}^{n+1}_+$. We compute the first and the second variation of this functional, and we derive the stationary conditions and a stability inequality for smooth minimizers. Also in this case, in order to estimate the Hausdorff dimension of the singular set of a smooth minimizer, it can be proved that is sufficient to look at smooth cones with an isolated singularity.

We present the results obtained by Edelen, Chodosh, and Li in [2]. Here, the non-existence of smooth minimizing cones depends not only on the dimension, but also on the contact angle θ. Again, the strategy is to prove that any smooth cone with an isolated singularity that minimizes $\mathcal{A}^\theta$ satisfies $|A| = 0$.

We prove, in dimension $n = 3$, that for any contact angle θ, there are no smooth minimizing cones with an isolated singularity. The arguments in this dimension use the stability inequality, like in the proof of Simons' Theorem. However, here is crucial the use of Gauss-Bonnet theorem and complex analysis, that are applicable only because the intersection of a cone with S^3 has dimension 2.

Also in dimension $n \leq 6$, we prove the non-existence of smooth minimizing cones through the stability inequality. But in this case, we have the additional

constraint of choosing angles $\theta \in (\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon)$, where ϵ is a small constant depending on the dimension n only.

Finally, in dimension $n = 4$, we deal with contact angles θ close to 0. It is interesting how, in this case, Edelen, Chodosh, and Li, don't use the stability inequality. Instead, they find a connection, for small angles, between minimizers of $\mathcal{A}^\theta$ in $\mathbb{R}^5$, and minimizers of the Alt-Caffarelli functional in $\mathbb{R}^4$. They prove that, for small angles θ, any smooth cone minimizing $\mathcal{A}^\theta$ can be written as a graph of a Lipschitz function over $\mathbb{R}^n$. They prove also that, for any sequence Ω_i of smooth cones minimizing $\mathcal{A}^{\theta_i}$, and $u_i : \mathbb{R}^n \to \mathbb{R}$ being the Lipschitz functions such that $\partial\Omega_i \cap \mathbb{R}^{n+1}_+ = \text{graph}(u_i)\llcorner\{u > 0\}$, the rescaled $\frac{u_i}{\tan(\theta_i)}$ converges to a one-homogeneus minimizer of the Alt-Caffarelli functional in $\mathbb{R}^4$. Thanks to the work of Jerison and Savin in [3], the only one-homogeneus minimizers of the Alt-Caffarelli functional in dimension 4 are flat solutions. Using this, Edelen, Chodosh, and Li prove that, for small angles, also smooth minimizing cones of $\mathcal{A}^\theta$ are flat.

In [3], Jerison and Savin emulate in some sense the techniques used in dealing with smooth cones with an isolated singularity minimizing some functional. Namely, they have a stability inequality, they have stationary conditions, and they have an interior inequality similar to the Simons' inequality. Also in their case there is a boundary term, as well as in the stability inequality for $\mathcal{A}^\theta$(see (2.21)). They provide a boundary inequality in order to deal with their boundary term. They then use a competitor in the stability inequality depending on a power of $|\nabla^2 u|$ in dimension $n = 3$, and depending on a function of the eigenvalues of $\nabla^2 u$ in dimension $n = 4$. Here u is a one-homogeneus minimizer, and $|\nabla^2 u|$ can be thought as the analogous of $|A|$.

Since this way to deal with the problem seemed more natural to us, we adapted their ideas to the context of $\mathcal{A}^\theta$. Following this path, we were able to provide a new proof of the non-existence of smooth cones minimizers of $\mathcal{A}^\theta$ in $\mathbb{R}^{n+1}_+$, when $n = 3$. If we assume that the smooth cone is the region below a graph of a Lipschitz function, making the comparison with the work of Jerison and Savin, we think that deal with the case $n = 4$ is just a matter of computation. Namely, we propose a competitor that should work in dimension $n = 4$, and we refer to [3] for a proof.

CHAPTER 1

Regularity of perimeter minimizers

1.1 Definitions and basic properties

Here we define sets of locally finite perimeter and some of their properties. Our main reference for this Chapter is [1].

Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. We say that E is a set of locally finite perimeter if the real valued functional

$$T \to \int_E \mathrm{div} T, \qquad T \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$$

is bounded, with respect to the uniform norm on $C_c^\infty(B(0,R);\mathbb{R}^n)$, for any $R > 0$. Using the Riesz representation Theorem, it is immediate to see that E has locally finite perimeter if and only if there exists a unique $\mathbb{R}^n$-valued Radon measure μ_E such that

$$\int_E \mathrm{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E, \qquad \forall T \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n).$$

We say that μ_E is the Gauss-Green measure of E, and we call its total variation $|\mu_E|$ the perimeter measure of E, that will be denoted by

$$P(E;F) = |\mu_E|(F), \qquad P(E) = |\mu_E|(\mathbb{R}^n),$$

for any $F \subset \mathbb{R}^n$. We call $P(E)$ the perimeter of E, and $P(E; F)$ the relative perimeter of E in F. If E is an open set with C^1-boundary, the divergence theorem says that $\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial E$, where ν_E is the outer unit normal of E, and $\mathcal{H}^{n-1}$ is the $(n-1)-$dimensional Hausdorff measure. For a general set of locally finite perimeter, we always have

$$\mathrm{spt}(\mu_E) \subset \partial E,$$

but we can define a unit normal only on a portion of the boundary.

Definition 1.1.1. The reduced boundary of E, that we denote $\partial^\star E$, is the set of those $x \in \mathrm{spt}\mu_E$ such that the limit $\lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E|(B(x,r))} =: \nu_E(x)$ exists and belongs to $\mathcal{S}^{n-1}$. We call ν_E the measure-theoretic outer unit normal to E.

As one would expect, sets of locally finite perimeter in $\mathbb{R}$ are not of interest, as they are just a countable union of intervals.

Proposition 1.1.2. *A Lebesgue measurable set $E \subset \mathbb{R}$ is of locally finite perimeter if and only if it is equivalent to a countable union of open intervals lying at mutually positive distance.*

In the previous statement we used the following definition.

Definition 1.1.3. Let E and E' two Lebesgue measurable sets. We say that E is equivalent to E' if

$$|E \Delta E'| = 0.$$

Remark 1.1.4. Let E and E' be two equivalent sets. Then, E is of locally finite perimeter if and only if E' is of locally finite perimeter. In this case,

$$\mu_E = \mu_{E'}.$$

There exists a powerful carachterization of μ_E and ν_E that is given by the De Giorgi's structure theory. We summarize it here with the next theorem.

Theorem 1.1.5. *If E is a set of locally finite perimeter, $\partial^\star E$ is $(n-1)$-rectifiable, and we have*

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^\star E, \qquad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^\star E.$$

Moreover, if $x \in \partial^\star E$, then $\nu_E(x)$ is orthogonal to $\partial^\star E$ at x, in the sense that there is the weakly-convergence of measures*

$$\mathcal{H}^{n-1} \llcorner \left(\frac{\partial^\star E - x}{r} \right) \overset{\star}{\rightharpoonup} \mathcal{H}^{n-1} \llcorner \nu_E(x)^\perp \qquad \text{as } r \to 0^+,$$

and it is outer to E at x, in the sense that there is the local convergence of sets

$$\frac{E-x}{r} \overset{loc}{\longrightarrow} \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \leq 0\} \qquad \text{as } r \to 0^+.$$

In the previous Theorem, by local convergence of a sequence of sets $E_i \overset{loc}{\longrightarrow} F$, we mean that the sequence $(\mathbb{1}_{E_i})_{i\in\mathbb{N}}$ converges to $\mathbb{1}_F$ in $L^1_{loc}(\mathbb{R}^n)$. An important property of sets of locally finite perimeter is the lower semicontinuity of perimeter.

Proposition 1.1.6. *If $\{E_h\}$ is a sequence of sets of locally finite perimeter in $\mathbb{R}^n$, with*

$$E_h \overset{loc}{\to} E, \qquad \limsup_{h\to\infty} P(E_h; K) < \infty,$$

for any compact set K in $\mathbb{R}^n$, then E is of locally finite perimeter in $\mathbb{R}^n$, $\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E$ and, for every open set $A \subset \mathbb{R}^n$, we have

$$P(E; A) \leq \liminf_{h\to\infty} P(E_h; A).$$

Sets of locally finite perimeter are closed for union and intersection, and satisfy a compactness theorem. Moreover, the intersection with an half-space decreases the perimeter.

Lemma 1.1.7. *If E and F are sets of locally finite perimeter in $\mathbb{R}^n$, then so are $E \cup F$ and $E \cap F$, and, for $A \subset \mathbb{R}^n$ open,*

$$P(E \cup F; A) + P(E \cap F; A) \leq P(E; A) + P(F; A).$$

Lemma 1.1.8. *Let E be a set of finite perimeter and let $e \in S^n$. Then, for almost every $t \in \mathbb{R}$, $E \cap \{x \cdot e < t\}$ is a set of finite perimeter, and*

$$P(E \cap \{x \cdot e < t\}) \leq P(E)$$

Theorem 1.1.9. *If $\{E_h\}_{h\in\mathbb{N}}$ are sets of locally finite perimeter in $\mathbb{R}^n$ with*

$$\sup_{h\in N} P(E_h; B_R) < \infty, \qquad \forall R > 0,$$

then there exists E of locally finite perimeter such that, up to a subsequence

$$E_h \overset{loc}{\to} E, \qquad \mu_{E_h} \overset{*}{\rightharpoonup} \mu_E.$$

Notice that this version of the compactness theorem gives only the weak* convergence of the measures μ_E, and don't imply the convergence of the total variations $|\mu_E|$.

We give now the notion of perimeter minimizer.

Definition 1.1.10 (Perimeter minimizer)**.** Let E be a set of locally finite perimeter in $\mathbb{R}^n$ that satisfies $\mathrm{spt}(\mu_E) = \partial E$, $n \geq 2$, and let A be an open set. We say that E is a perimeter minimizer in A if, for any $r > 0$, any $x \in \mathbb{R}^n$, and any set of locally finite perimeter F that sarisfies $E\Delta F \subset\subset B(x,r) \cap A$, the following inequality holds

$$P(E; B(x,r)) \leq P(F; B(x,r)).$$

If $A = \mathbb{R}^n$, we say that E is a global perimeter minimizer, or just a perimeter minimizer.

Remark 1.1.11. The assumption $\mathrm{spt}(\mu_E) = \partial E$ is not restrictive. Indeed, it can be proven that, for any set of locally finite perimeter E, there exists a set of locally finite perimeter E' that is equivalent to E, and such that

$$\partial E' = \mathrm{spt}(\mu_E) = \mathrm{spt}(\mu_{E'}).$$

1.2 First and Second Variation of Perimeter

If E is a perimeter minimizer in an open set A, the perimeter decreases under small perturbations of E in a compact subset of A. We can perform the perturbations continuously in the time t, and then take first and second derivative in t in order to obtain stationarity and stability conditions for E.

Definition 1.2.1. Let $A \subset \mathbb{R}^n$ be an open set, let $\epsilon > 0$, and let $(f_t)_{|t|<\epsilon}$ be a one parameter family of diffeomorphisms. We say that $(f_t)_{|t|<\epsilon}$ is a local variation in A if there exists a compact subset K of A such that

$$\begin{aligned} &f_0(x) = x, \qquad \forall\, x \in \mathbb{R}^n, \\ &\{x \in \mathbb{R}^n : f_t(x) \neq x\} \subset K, \qquad \forall |t| < \epsilon. \end{aligned}$$

If $(f_t)_{|t|<\epsilon}$ is a local variation in A, we say that $T(x) := \frac{\partial f_t}{\partial t}(x,0)$ is the initial velocity of $(f_t)_{|t|<\epsilon}$.

Remark 1.2.2. If $(f_t)_{|t|<\epsilon}$ is a local variation in A, its initial velocity T has compact supoport in A, $f_t(A) = A$, and, for any $E \subset \mathbb{R}^n$, $f_t(E)\Delta E \subset\subset A$. Moreover, the following Taylor's expansions holds uniformly in $x \in \mathbb{R}^n$,

$$f_t(x) = x + tT(x) + O(t^2), \qquad \nabla f_t(x) = Id + t\nabla T(x) + O(t^2), \tag{1.1}$$

where with $\nabla T(x)$ we denote the matrix representing the differential of T in x with respect to the canonical basis of $\mathbb{R}^n$.

We say that a set of locally finite perimeter E is stationary for perimeter in an open bounded set A if $\mathrm{spt}\mu_E = \partial E$ and

$$\frac{d}{dt}P(f_t(E);A)_{|t=0} = 0, \tag{1.2}$$

for any $(f_t)_{|t|<\epsilon}$ local variation in A. If, for any local variation in A, holds also

$$\frac{d^2}{dt^2}P(f_t(E);A)_{|t=0} \geq 0, \tag{1.3}$$

we say that E is stable for perimeter in A.

Remark 1.2.3. If E is a perimeter minimizer in A, and $(f_t)_{|t|<\epsilon}$ is a local variation in A, since $f_t(E)\Delta E \subset\subset A$, we have that $P(E;A) \leq P(f_t(E);A)$. Thus, if we knew that any path $P(f_t(E);A)$ were regular enough we could conclude that E is stationary and stable for the perimeter in A. We will prove this in the case of E open set with smooth boundary.

Conversely, if we start from $T \in C_c^\infty(A)$, there are two canonycal ways to construct a local variation that has T has initial velocity.

The first method consists of setting

$$f_t(x) = x + tT(x),$$

tho only non trivial property to check is that f_t is a diffeomorphism for t small enough. That follows by using the inverse function theorem and the Neumann series. Through this chapter we will use only the previous method, but later we will need also to consider a local variation given by the flow of an ODE:

$$\begin{aligned} &\frac{\partial}{\partial t}f(t,x) = T(f(t,x)),\\ &f(0,x) = x. \end{aligned}$$

In order to compute the first and the second variation of the perimeter we need a Lemma first.

Lemma 1.2.4. *Let Z be a real valued $n \times n$-matrix. Then*

$$\begin{aligned} &(Id + tZ)^{-1} = Id - tZ + t^2Z^2 + O(t^3),\\ &\det(Id + tZ) = 1 + t\mathrm{Tr}(Z) + \frac{t^2}{2}(\mathrm{Tr}(Z)^2 - \mathrm{Tr}(Z^2)) + O(t^3) \end{aligned}$$

Proof. The first equation follows since, for t small enough, the Neumann series gives $(Id + tZ)^{-1} = \sum_{i\in\mathbb{N}} (-tZ)^i$.

In order to prove the second equation, let $(\lambda_i)_{i=1}^n$ the complex eigenvalues of Z, and let $P \in GL(n, \mathbb{C})$ such that $Y := PZP^{-1}$ is upper triangular. Thus

$$\det(Id + tZ) = \prod_{i=1}^{n}(1 + t\lambda_i) = 1 + t\sum_{i=1}^{n}\lambda_i + t^2\sum_{i<j}\lambda_i\lambda_j + O(t^3).$$

Notice now that $\sum_{i=1}^{n} \lambda_i = \mathrm{Tr}(Z)$, and that

$$\sum_{i<j}\lambda_i\lambda_j = \frac{1}{2}\sum_{i,j=1}^{n}\lambda_i\lambda_j - \frac{1}{2}\sum_{i=1}^{n}\lambda_i^2 = \frac{1}{2}(\mathrm{Tr}(Z)^2 - \mathrm{Tr}(Y^2)),$$

and $\mathrm{Tr}(Y^2) = \mathrm{Tr}(PZ^2P^{-1}) = \mathrm{Tr}(Z^2)$, ending the proof. □

We are now ready to compute the first variation of perimeter.

Theorem 1.2.5 (First variation of perimeter)**.** *Let A be an open bounded set in $\mathbb{R}^n$, E be an open set with smooth boundary in A, and $(f_t)_{|t|<\epsilon}$ be a local variation in A with initial velocity T. Then,*

$$P(f_t(E); A) = P(E; A) + t\int_{\partial E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} + O(t^2), \tag{1.4}$$

where $\mathrm{div}_E T(x) := \mathrm{div}_T(x) - \nu_E(x)\cdot\nabla T(x)\nu_E(x)$ *is the boundary divergence of T on E.*

Proof. Let us call $g_t := f_t^{-1}$. Notice that $f_t(E)$ is an open set with boundary $\partial f_t(E) = f_t(\partial E)$ that is smooth in $A = f_t(A)$, and with outer normal given by

$$\nu_{f_t(E)}(x) = \frac{\Big(\nabla g_t(x)\Big)^{\star}\nu_E(g_t(x))}{\Big|\Big(\nabla g_t(x)\Big)^{\star}\nu_E(g_t(x))\Big|}.$$

Thus, by the tangential area formula,

$$P(f_t(E); A) = \int_{f_t(\partial E\cap A)} 1\, d\mathcal{H}^{n-1} = \int_{\partial E\cap A} \mathrm{J}_E f_t(x)\, d\mathcal{H}^{n-1}(x),$$

where $\mathrm{J}_E f_t(x)$ is the tangential jacobian on ∂E, given by $\mathrm{J}_E f_t(x) := |\det(A)|$, where A is the $(n-1)\times(n-1)$-matrix that represents the linear application $df_t(x) : T_{\partial E}(x) \to T_{f_t(\partial E)}(x)$ with respect to any orthonormal basis of $T_{\partial E}(x)$ and any orthonormal basis of $T_{f_t(\partial E)}(x)$. Fix such two orthonormal basis and complete them, with $\nu_E(x)$ and $\nu_{f(E)}(x)$, to orthonormal basis of $\mathbb{R}^n$. Thus, in those basis, $df_t(x) : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\left(\begin{array}{c|c} \mathbf{A} & \begin{matrix} * \\ * \end{matrix} \\ \hline 0\dots0 & \nabla f_t(x)\nu_E(x)\cdot\frac{\nabla f_t(x)^{-*}\nu_E(x)}{|\nabla f_t(x)^{-*}\nu_E(x)|} \end{array}\right) =$$

$$= \left(\begin{array}{c|c} \mathbf{A} & \begin{matrix} * \\ * \end{matrix} \\ \hline 0\dots0 & |\nabla f_t(x)^{-*}\nu_E(x)|^{-1} \end{array}\right),$$

that means $\mathrm{J}f_t(x)|\nabla f_t(x)^{-*}\nu_E(x)| = \mathrm{J}_E f_t(x)$.

Using (1.1) and Lemma 1.2.4 we can write

$$\mathrm{J}f_t(x) = \det(Id + t(\nabla T(x) + O(t))) = 1 + t\mathrm{div}T + O(t^2),$$

$$\begin{aligned} |\nabla f_t(x)^{-*}\nu_E(x)| &= |(Id + t(\nabla T^* + O(t)))^{-1}\nu_E(x)| \\ &= |\nu_E(x) - t\nabla T^*\nu_E(x) + O(t^2)| \\ &= 1 - \nu_E(x)\cdot\nabla T^*\nu_E(x) + O(t^2) \\ &= 1 - t\nu_E(x)\cdot\nabla T\nu_E(x) + O(t^2), \end{aligned}$$

where we used that $\nu_E(x) = 1$, and that the estimates are uniform in $x \in \mathbb{R}^n$. Putting the previous equations together we get

$$\mathrm{J}_E f_t(x) = 1 + t\mathrm{div}_E T(x) + O(t^2),$$

and an integration over $\partial E \cap A$ ends the proof. □

Remark 1.2.6. We could have stated the previous theorem for a generic E of locally finite perimeter, and the proof is the same, but we would have needed an area formula for set of locally finite perimeter. However, since we are going to work only with sets that are regular up to a point, the statement above is sufficient for our purpose.

We will make use in particular of the following Corollary:

Corollary 1.2.7. *Let E be an open set with smooth boundary. If E is a perimeter minimizer in the open set A, then*

$$H(x) = 0, \qquad \forall x \in \partial E \cap A, \tag{1.5}$$

where H is the mean curvature of ∂E.

Proof. Take a smooth vector field T that has compact support in A, let $R > 0$ such that $\mathrm{spt}(T) \subset B(0,R) \cap A$, and let $(f_t)_{|t|<\epsilon}$ be a local variation in $B(0,R) \cap A$ with initial velocity T. Thanks to Theorem 1.2.5 and Remark 1.2.3 this means

$$\int_{\partial E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = 0,$$

and the tangential divergence Theorem implies

$$\int_{\partial E} H\nu_E \cdot T \, d\mathcal{H}^{n-1} = 0,$$

thus $H = 0$. □

We are going now to compute the second variation of perimeter. Although our assumption of smoothness was not necessary in Theorem 1.2.5, now we want to take dervatives of the outer normal of a set of locally finite perimeter E, therefore the request of some regularity on the boundary is not removable.

Let E be an open set with smooth boundary in an open set $A \subset \mathbb{R}^n$. By the existence of a tubular neighbourhood of $\partial E \cap A$ in A, we can deduce that there is an open set A' with $A \cap \partial E \subset A' \subset A$ such that the signed distance function $s_E : \mathbb{R}^n \to \mathbb{R}$ of E

$$s_E(x) := \begin{cases} \mathrm{dist}(x, \partial E), & x \in \mathbb{R}^n \setminus E, \\ -\mathrm{dist}(x, \partial E), & x \in E, \end{cases}$$

satisfies $s_E \in C^\infty(A')$. Let us define

$$N_E = \nabla s_E, \qquad A_E = \nabla^2 s_E, \qquad \text{on } A'.$$

Notice that N_E is an extension of the outer unit normal ν_E and satisfies

$|N_E| \equiv 1$, while A_E is a symmetric matrix that we call the second fundamental form of ∂E. Notice that the trace of A_E is the mean curvature of ∂E, and that the squared norm of the second fundamental form $|A|^2 = \sum_{i,j} A_{ij}^2$ is invariant under change of coordinates, since $|A|^2 = \mathrm{Tr}(A^2)$.

We compute now the second variation of perimeter under local variations with initial velocity proportional to N_E.

Theorem 1.2.8 (Second variation of perimeter). *Let E be an open set with smooth boundary in the open set A, let $B \subset A$ be open and bounded, and let $\zeta \in C_c^\infty(B)$. If $(f_t)_{|t|<\epsilon}$ is a local variation with initial velocity $T = \zeta N_E \in C_c^\infty(B;\mathbb{R}^n)$, then*

$$\frac{d^2}{dt^2} P(f_t(E);B)_{|t=0} = \int_{\partial_E} |\nabla_E \zeta|^2 + (H_E^2 - |A_E|^2)\zeta^2 \, d\mathcal{H}^{n-1}. \tag{1.6}$$

In particular, if E is a perimeter minimizer in A, then

$$\int_{\partial_E} |\nabla_E \zeta|^2 - |A_E|^2 \zeta^2 \, d\mathcal{H}^{n-1} \geq 0, \tag{1.7}$$

Proof. As in the proof of Theorem 1.2.5, we have

$$P(f_t(E);B) = \int_{B\cap\partial E} \mathrm{J} f_t |\nabla f_t^{-*} \nu_E| \, d\mathcal{H}^{n-1}, \tag{1.8}$$

and, by the Theorem of differentiation under the integral sign, we get that the function $t \to P(f_t(E);B)$ is smooth in a neighbourhood of zero. In order to ease the notation we assume that $f_t(x) = x + Tx$. The following relations hold

$$\begin{aligned}
&\nabla T = \zeta A_E + N_E \otimes \nabla\zeta,\\
&(\nabla T)^2 = \zeta^2 A_E^2 + \zeta(N_E \otimes \nabla\zeta)A_E + (N_E\cdot\nabla\zeta)N_E \otimes \nabla\zeta,\\
&\mathrm{Tr}(\nabla T) = \zeta H_E + N_E \cdot \nabla\zeta,\\
&\mathrm{Tr}((\nabla T)^2) = \zeta^2 |A_E|^2 + (N_E\cdot\nabla\zeta)^2
\end{aligned}$$

where we used the notation $v \otimes w := vw^\star$ for any $v, w \in \mathbb{R}^n$, we used the relations

$$\begin{aligned}
&A_E(N_E \otimes w) = (A_E N_E) \otimes w = 0 \otimes w = 0,\\
&(v\otimes w)^2 = (vw^\star)(vw^\star) = v(w^\star v)w^\star = (v\cdot w)v \otimes w,\\
&\mathrm{Tr}(v\otimes w) = \sum_i v_i w_i = v \cdot w,
\end{aligned}$$

and we used that A_E is symmetric in computing

$$\begin{aligned}
&(N_E\cdot\nabla\zeta)A_E = N_E \nabla\zeta^\star A_E = N_E (A_E\nabla\zeta)^\star,\\
&\mathrm{Tr}((N_E\cdot\nabla\zeta)A_E) = N_E\cdot(A_E\nabla\zeta) = N_E^\star A_E \nabla\zeta = (A_E N_E)^\star \nabla\zeta = 0\cdot\nabla\zeta = 0.
\end{aligned}$$

Now, by using Lemma 1.2.4, we can write

$$\begin{aligned}\mathrm{J}f_t &= \det(Id + t\nabla T) = 1 + t\mathrm{Tr}(\nabla T) + \frac{t^2}{2}(\mathrm{Tr}(\nabla T)^2 - \mathrm{Tr}(\nabla T^2)) + O(t^3)\\ &= 1 + t(\zeta H_E + N_E \cdot \nabla\zeta) + \frac{t^2}{2}(\zeta^2 H_E^2 - \zeta^2|A_E|^2 + 2\zeta H_E N_E \cdot \nabla\zeta) + O(t^3).\end{aligned}$$

Concerning the other factor in $\mathrm{J}_E f_t$,

$$\begin{aligned}(\nabla f_t)^{-\star} N_E &= (Id + t(\nabla T)^\star)^{-1} N_E =\\ &= N_E - t\nabla T^\star N_E + t^2(\nabla T^\star)^2 N_E + O(t^3).\end{aligned}$$

Since $|N_E| = 1$, if we call $\gamma(t) = (\nabla f_t)^{-\star} N_E$, we can compute

$$\begin{aligned}|\gamma(t)| &= 1 + \gamma(0)\cdot\gamma'(0)t + \left(\frac{\gamma(t)}{|\gamma(t)|}\cdot\gamma'(t)\right)'(0)\frac{t^2}{2} + O(t^3)\\ &= 1 - tN_E \cdot \nabla T^\star N_E +\\ &+ \frac{t^2}{2}(2N_E \cdot (\nabla T^\star)^2 N_E + |\nabla T^\star N_E|^2 + -(N_E \cdot \nabla T^\star N_E)^2) + O(t^3).\end{aligned}$$

We can make more explicit this espression using the previous relations:

$$\begin{aligned}&\nabla T^\star N_E = \nabla\zeta,\\ &N_E \cdot (\nabla T^\star)^2 N_E = \zeta N_E \cdot A_E \nabla\zeta + (\nabla\zeta \cdot N_E)^2\\ &= (\nabla\zeta \cdot N_E)^2,\\ &|\nabla_E\zeta|^2 = |\nabla\zeta|^2 - |\nabla\zeta \cdot N_E|^2.\end{aligned}$$

Therefore, we can finally multiply the develops of $\mathrm{J}f_t$ and of $|(\nabla f_t)^{-\star} N_E|$,

$$\begin{aligned}\mathrm{J}f_t|(\nabla f_t)^{-\star} N_E| &= 1 + t\zeta H_E +\\ &+ \frac{t^2}{2}(|\nabla_E\zeta|^2 + \zeta^2(H_E^2 - |A_E|^2)) + O(t^3),\end{aligned}$$

and, since this develop is uniform in $x \in \mathbb{R}^n$, an integration ends the proof. □

1.3 Analysis of cones

We first recall the definition of cone.

Definition 1.3.1. An open set $\Omega \subset \mathbb{R}^n$ is a cone with vertex at 0 if , for any $x \in \Omega$ and any $\lambda > 0$, $\lambda x \in \Omega$. We say that the cone is smooth if it has smooth boundary in $\mathbb{R}^n \setminus 0$.

The first part of this section is devoted to explain why we are interested in studying cones.

It can be proven the following powerful regularity Theorem for perimeter minimizers

Theorem 1.3.2. *If $A \subset \mathbb{R}^n$ is an open set, and E is a perimeter minimizer in A , then $A \cap \partial^* E$ is a smooth hypersurface.*

What is left of the boundary of E is the so-called *singular set*

$$\Sigma(E; A) := A \cap (\partial E \setminus \partial^* E).$$

Definition 1.3.3. If a cone $\Omega \subset \mathbb{R}^n$ is a perimeter minimizer such that

$$\Sigma(\Omega) := \Sigma(\Omega; \mathbb{R}^n) \neq \emptyset,$$

we say that Ω is a singular minimizing cone.

If

$$\Sigma(\Omega) = \{0\},$$

we say that Ω is a smooth minimizing cone.

The analysis of the singular set is related to the study of singular minimizing cones in $\mathbb{R}^n$. Indeed, whenever x is a singular point, the rescalings $\frac{E-x}{r}$ of E at x converge, as $r \to 0$, to a cone K, which is singular at 0, and is a perimeter minimizer in $\mathbb{R}^n$.

Theorem 1.3.4. *Let E be a perimeter minimizer in an open set $A \subset \mathbb{R}^n$, let $x \in \Sigma(E; A)$, and let, for $r > 0$, $E_{x,r} := \frac{E-x}{r}$.*

Then there exists a singular minimizing cone $\Omega \subset \mathbb{R}^n$, such that, up to a subsequence

$$E_{x,r} \overset{loc}{\to} \Omega, \qquad \mu_{E_{x,r}} \overset{*}{\rightharpoonup} \mu_\Omega, \qquad |\mu_{E_{x,r}}| \overset{*}{\rightharpoonup} |\mu_\Omega|, \qquad \text{as}\, r \to 0.$$

A singular minimizing cone may have more singular points than 0. This issue is solved by the so called *Federer's dimension reduction principle.*

Theorem 1.3.5 (Dimension reduction principle)**.** *Let Ω be a singular minimizing cone in $\mathbb{R}^n$, and let $x_0 \in \Sigma(K)$, $x_0 \neq 0$. Then, up to a subsequence and up to a rotation,*

$$\Omega_{x_0,r} \overset{loc}{\to} F \times \mathbb{R}, \qquad \text{as}\, r \to 0,$$

where F is a singular minimizing cone in $\mathbb{R}^{n-1}$.

We need also a Lemma that gives a lower bound on the dimensions in which there could exist a singular cone Ω with $\Sigma(\Omega) \neq \{0\}$.

Lemma 1.3.6. *If Ω is a singular minimizing cone in $\mathbb{R}^n$, $x_0 \in \Sigma(\Omega)$, and $x_0 \neq 0$, then $n \geq 3$.*

The last Theorems and the last Lemma allow to relate the Hausdorff dimension of the singular sets to the analysis of the perimeter minimizing cones in $\mathbb{R}^n$. Precisely, Federer proved the following Theorem

Theorem 1.3.7. *There is a critical dimension n^* (defined as the first dimension n in which there is a smooth minimizing cone), such that*

- *if $n < n^*$, then the singular set $\Sigma(E; A)$ is empty for any perimeter minimizer E in an open set $A \subset \mathbb{R}^n$;*
- *if $n = n^*$, then for any perimeter minimizer E in an open set $A \subset \mathbb{R}^n$, the singular set $\Sigma(E; A)$ is a discrete set of points;*
- *if $n > n^*$, then for any perimeter minimizer E in an open set $A \subset \mathbb{R}^n$, the Hausdorff dimension of $\Sigma(E; A)$ is at most $n - n^*$, that is:*

$$\mathcal{H}^{n-n^*+\varepsilon}(\Sigma(E; A)) = 0 \quad \textit{for every} \quad \varepsilon > 0.$$

Proof. We prove just the first part of the statement. If $n^* = 1, 2$ there is nothing to prove. If $2 \leq n < n^*$, let $E \subset \mathbb{R}^n$ a perimeter minimizer in some open set A. If there exists $x \in \Sigma(E; A)$, then, by Theorem 1.3.4, there exists a singular minimizing cone Ω in $\mathbb{R}^n$. Since $n < n^*$, there exists $x \in \Sigma(\Omega) \setminus \{0\}$. Then, by Lemma 1.3.6, $n \geq 3$. By Theorem 1.3.5, there exists a singular minimizing cone in $\mathbb{R}^{n-1}$. Since $2 \leq n-1 < n^*$, we can iterate this construction a finite number of times, getting $n \geq n^*$, that is a contradiction. $\square$

For the rest of this section Ω will be always a smooth cone.

It is known that if M is a Riemannian manifold and $p \in M$, there exists a coordinate system around p such that the metric is the identity at the first order in p. In order to obtain more manageable expressions for the objects that we want to deal with, we will describe an explicit such coordinate system. If $x_0 \in \partial\Omega \setminus \{0\}$, up to a rotation and a translation, we can assume that x_0 is proportional to e_1 and that, locally in x_0, Ω is the region above the graph of a smooth function $u : \mathbb{R}^{n-1} \supset U \to \mathbb{R}$ such that $u(x_0) = 0$, $\nabla u(x_0) = 0$, $\nabla^2 u(x_0)$ is dyagonal and such that u is 1-homogeneus in the e_1-direction at x_0, where e_1 is the first vector of the canonical basis of $\mathbb{R}^n$.

Lemma 1.3.8. *Let* Ω *and* u *as above. Then*

$$\nu_\Omega(x, u(x)) = \frac{(\nabla u(x), -1)}{\sqrt{1 + |\nabla u(x)|^2}},$$
$$g(x, u(x)) = Id + \nabla u \otimes \nabla u,$$
$$A = \frac{1}{\sqrt{1 + |\nabla u|^2}} g^{-1} \nabla^2 u,$$
$$\partial_i A \circ \psi(x_0, 0) = \nabla^2 u_i(x_0), \qquad i \leq n$$
$$\partial_i \partial_j A \circ \psi(x_0, 0) = -\nabla u_i \cdot \nabla u_j \nabla^2 u(x_0) - (\nabla u_i \otimes \nabla u_j + \nabla u_j \otimes \nabla u_i) \nabla^2 u(x_0) + \nabla^2 u_{ij}(x_0), \qquad i \leq n$$

in particular

$$H_\Omega(x_0) = \Delta u(0),$$
$$|A_\Omega|^2(x_0) = |\nabla^2 u(0)|^2.$$

Proof. u induces a parametrization of $\partial\Omega$ given by $\psi(x) := (x, u(x))$, thus $T_{\psi(x)}\partial\Omega = \text{Span}\{(e_i, u_i(x))\} = (\nabla u(x), -1)^\perp$, and the sign of the outer normal is fixed provided that Ω is the region above the graph of u. Since $g[\psi_i, \psi_j] = \psi_i \cdot \psi_j = \delta_{ij} + u_i u_j$, $g = Id + \nabla u \otimes \nabla u$. Let $\tilde{A}$ be the matrix representing, in the coordinates induced by ψ, the scalar product induced by the symmetric endomorphism $d\nu_\Omega(\psi(x)) : T_x\partial\Omega \to T_x\partial\Omega$. It is well known that $A = g^{-1}\tilde{A}$, and

$$\begin{aligned}
\psi_i \cdot \tilde{A}\psi_j &= \psi_i \cdot (\nu_\Omega \circ \psi)_j \\
&= (e_i, u_i) \cdot \Big(\frac{(\nabla u_j, 0)}{\sqrt{1 + |\nabla u|^2}} - \nabla u_j \cdot \nabla u \frac{(\nabla u, -1)}{(1 + |\nabla u|^2)^{3/2}}\Big) \\
&= \frac{u_{ij}}{\sqrt{1 + |\nabla u|^2}},
\end{aligned}$$

thus $A = g^{-1} \nabla^2 u \frac{1}{\sqrt{1+|\nabla u|^2}}$. Moreover, by the Neumann series, and taking into account that $\nabla u(x_0) = 0$,

$$g^{-1} = Id - \nabla u \otimes \nabla u + |\nabla u|^2 \nabla u \otimes \nabla u + o(|\nabla u|^4),$$

noticing that first and second derivatives of $o(|\nabla u|^2(x))$ at x_0 are 0. Finally,

$$\partial_i \partial_j \frac{1}{\sqrt{1 + |\nabla u|^2}}(x_0) = -\partial_i(\nabla u_j \cdot \nabla u (1 + |\nabla u|^2)^{-3/2})(x_0) = -\nabla u_j \cdot \nabla u_i.$$

Taking the derivatives of A at x_0 and taking into account the previous estimates end the proof. □

Theorem 1.3.9 (Simons inequality)**.** *Let Ω be an open cone with zero mean curvature. Then,*

$$|A|^4 - |\nabla_{\partial\Omega}|A||^2 + \frac{1}{2}\Delta_{\partial\Omega}|A|^2 \geq \frac{2|A|^2}{|x|^2}, \tag{1.9}$$

whenever $|A|(x_0) \neq 0$

Proof. Take x_0 and u as above. Notice that it is sufficient to prove the thesis at x_0 under the coordinates induced by u. Computing and using that A is symmetric,

$$\nabla_{\partial\Omega}|A|^2(x, u(x)) = \sum_{i=1}^{n} \partial_i(|A|^2 \circ \psi)(x)(e_i, u_i(x))(1 + |u_i(x)|^2)^{-1},$$

thus,

$$\nabla_{\partial\Omega}|A|^2(x_0) = \nabla(|A|^2 \circ \psi)(x_0),$$

and

$$\begin{aligned}\Delta_{\partial\Omega}|A|^2(x_0) &= \sum_{j=1}^{n} \frac{1}{\sqrt{\det(g)}}\partial_j(\sqrt{\det(g\circ\psi)}\partial_j(|A|^2 \circ \psi)(1 + |u_j|^2)^{-1})(x_0)\\ &= \Delta(|A|^2 \circ \psi)(x_0),\end{aligned}$$

where we used that $u_i(x_0) = 0$, that $\partial_j(g \circ \psi)(x_0) = 0$, that $\partial_j(\sqrt{\det(g \circ \psi)})(x_0) = d_{Id}\sqrt{\det(\cdot)}[\partial_j(g \circ \psi)(x_0)]$, and that $\partial_j(1 + |u_j|^2)^{-1}(x_0) = -2u_j u_{jj}(x_0)(1 + |u_j|^2(x_0))^{-2}$.

Using the previous Lemma, we can compute

$$\begin{aligned}|\nabla_{\partial\Omega}|A||^2(x_0) &= \frac{1}{4|A|^2}|\nabla(|A|^2 \circ \psi)|^2(x_0)\\ &= \frac{1}{4|A|^2}\sum_{i=1}^{n} |\partial_i(\mathrm{Tr}(A^2 \circ \psi))|^2\\ &= \frac{1}{|A|^2}\sum_{i=1}^{n}(\mathrm{Tr}(A\partial_i A))^2(x_0)\\ &= \frac{1}{|A|^2}\sum_{i=1}^{n}(\mathrm{Tr}(\nabla^2 u \nabla^2 u_i))^2(x_0)\\ &= \frac{1}{|A|^2}\sum_{i=1}^{n}\left(\sum_{j,k=1}^{n} u_{jk}u_{ijk}\right)^2(x_0),\end{aligned}$$

and

$$\begin{aligned}\frac{\Delta(|A|^2 \circ \psi)}{2}(x_0) &= \sum_{i=1}^{n} \mathrm{Tr}(A\partial_{ii}(A \circ \psi)) + |\partial_i(A \circ \psi)|^2(x_0) \\ &= \sum_{i=1}^{n} \sum_{j,k=1}^{n} u_{ijk}^2(x_0) + \\ &- \sum_{i=1}^{n} u_{ii}^2 |\nabla^2 u|^2 - 2\mathrm{Tr}(\nabla u_i \otimes \nabla u_i (\nabla^2 u)^2) + \mathrm{Tr}(\nabla^2 u_{ii} \nabla^2 u).\end{aligned}$$

Taking into account that $\nabla^2 u(x_0)$ is diagonal, that $\nabla^2 u(x_0) = A(x_0)$, and that, for any two matrices B and C holds the equality $\mathrm{Tr}(BC) = \sum_{j,k=1}^{n} B_{jk} C_{kj}$, we can write

$$\begin{aligned}\frac{\Delta(|A|^2 \circ \psi)}{2}(x_0) = -|A|^4 - 2\sum_{i=1}^{n} u_{ii}^4 + \sum_{i,j=1}^{n} u_{iijj} u_{jj} + \\ + \sum_{i,j,k=1}^{n} u_{ijk}^2.\end{aligned}$$

Notice now that $\mathrm{Tr}(A \circ \psi) = H \circ \psi$ is identically zero on the domain of definition of ψ, and in particular, using again the previous Lemma and the fact that $\nabla^2 u(x_0)$ is diagonal, we get

$$\begin{aligned}0 &= \partial_j \partial_j \mathrm{Tr}(A \circ \psi)(x_0) = \mathrm{Tr}(\partial_j \partial_j (A \circ \psi))(x_0) \\ &= -|\nabla u_j|^2 \Delta u(x_0) - 2u_{jj}^3 + \Delta u_{jj} \\ &= -2u_{jj}^3 + \Delta u_{jj}.\end{aligned}$$

Using this relation, we obtain

$$\begin{aligned}-2\sum_{i=1}^{n} u_{ii}^4 + \sum_{i,j=1}^{n} u_{iijj} u_{jj} &= -2\sum_{i=1}^{n} u_{ii}^4 + \sum_{j=1}^{n} \Delta u_{jj} u_{jj} \\ &= -2\sum_{i=1}^{n} u_{ii}^4 + 2\sum_{j=1}^{n} u_{ii}^4 = 0,\end{aligned}$$

thus

$$\frac{\Delta(|A|^2 \circ \psi)(x_0)}{2} = -|A|^4 + \sum_{i,j,k=1}^{n} u_{ijk}^2.$$

Therefore, the left hand side of the (1.9) is equal to

$$\sum_{i,j,k=1}^{n} u_{ijk}^2 - \frac{1}{|A|^2}\sum_{i=1}^{n}\Big(\sum_{j,k=1}^{n} u_{jk}u_{ijk}\Big)^2(x_0) =$$
$$= |A|^{-2}\Big(\sum_{i,j,k,r,l=1}^{n} u_{ijk}^2 u_{rl}^2 - \sum_{i=1}^{n}\Big(\sum_{j,k=1}^{n} u_{jk}u_{ijk}\Big)^2\Big).$$

Notice now that, relabeling the indexes, holds

$$\sum_{i=1}^{n}\Big(\sum_{j,k=1}^{n} u_{jk}u_{ijk}\Big)^2 = \sum_{k=1}^{n}\Big(\sum_{i,j=1}^{n} u_{ij}u_{ijk}\Big)^2$$
$$= \sum_{k=1}^{n}\sum_{i,j=1}^{n} u_{ijk}u_{ij}\sum_{r,l=1}^{n} u_{rlk}u_{rl},$$

and a simple computation shows that

$$\sum_{i,j,k,r,l=1}^{n}(u_{ijk}^2u_{rl}^2 - u_{ijk}u_{ij}u_{rlk}u_{rl}) = \frac{1}{2}\sum_{i,j,k,r,l}(u_{rl}u_{ijk} - u_{ij}u_{rlk})^2.$$

We can then roughly estimate

$$\sum_{i,j,k,r,l}(u_{rl}u_{ijk} - u_{ij}u_{rlk})^2 \geq \sum_{k=1}^{n}\sum_{j,r,l=2}^{n}(u_{rl}u_{1jk} - u_{1j}u_{rlk})^2$$
$$+\sum_{k=1}^{n}\sum_{i,r,l=2}^{n}(u_{rl}u_{i1k} - u_{i1}u_{rlk})^2 + \sum_{k=1}^{n}\sum_{i,j,l=2}^{n}(u_{1l}u_{ijk} - u_{ij}u_{1lk})^2$$
$$+\sum_{k=1}^{n}\sum_{i,j,r=2}^{n}(u_{r1}u_{ijk} - u_{ij}u_{r1k})^2 = 4\sum_{k=1}^{n}\sum_{i,j,r=2}^{n}(u_{r1}u_{ijk} - u_{ij}u_{r1k})^2$$
$$= 4\sum_{k=1}^{n}\sum_{i,j,r=2}^{n} u_{ij}^2u_{r1k}^2,$$

where in the last inequality we used again that $\nabla^2 u(x_0)$ is diagonal. Moreover, ν_Ω is 0-homogeneus in direction e_1 at x_0, therefore $A(x_0)e_1 = 0$, and this means that $u_{i1}(x_0) = 0$, and thus $\sum_{i,j=2}^{n} u_{ij}^2 = |A|^2$. On the other hand, denoting by A_{ij} the coordinates of A, and being A $-1-$ homogeneus at x_0 in direction e_1,

$$u_{1ij}(x_0) = \partial_1(A_{ij}\circ\psi)(x_0) = -\frac{A_{ij}(x_0)}{|x_0|},$$

that ends the proof. □

We are finally ready to prove the celeber Simons' Theorem. (see [5])

Theorem 1.3.10 (Simons' Theorem)**.** *There are no smooth minimizing cones in $\mathbb{R}^n$, if $2 \le n \le 7$.*

Proof. Assume by contradiction that Ω is a smooth minimizing cone.

We will treat first the case $n = 2$. In this case, $\Omega \cap \mathcal{S}^1$ is a finite collection of disjoint circular arcs. Take one of these arcs, with endpoints x_0, x_1. If $x_0 \neq -x_1$, denote by T the closed triangle spanned by x_0, x_1 and 0, and consider $\Omega' := \Omega \setminus T$. Since in a triangle the lenght of a side is strictly smaller than the sum of the lenghts of the other two sides, we get that $P(\Omega'; B(0,2)) < P(\Omega; B(0,2))$, and $\Omega \Delta \Omega' \subset\subset B(0,2)$. Therefore, $\Omega \cap \mathcal{S}^1$ consists of just one circular arc with antipodal end-points, that is Ω is a half-plane.

Consider now the case of $n \ge 3$.

We will use the stability inequality to prove that the second fundamental form of $\partial\Omega \setminus \{0\}$ is identically zero. Since $\partial\Omega \setminus \{0\}$ is smooth, we are allowed to use, in (1.7), test functions with support away from zero. Fix $\epsilon > 0$, and take as a test function $\zeta_\epsilon := \varphi |A|_\epsilon$, where $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, and $|A|_\epsilon := \sqrt{|A|^2 + \epsilon^2}$. We have

$$\begin{aligned}
|\nabla_{\partial\Omega}\zeta_\epsilon|^2 &= \varphi^2 |\nabla_{\partial\Omega}|A|_\epsilon|^2 + |A|_\epsilon^2 |\nabla_{\partial\Omega}\varphi|^2 + \frac{1}{2}\nabla_{\partial\Omega}(\varphi^2) \cdot \nabla_{\partial\Omega}(|A|_\epsilon^2) \\
&= \varphi^2 |\nabla_{\partial\Omega}|A|_\epsilon|^2 + |A|_\epsilon^2 |\nabla_{\partial\Omega}\varphi|^2 + \frac{1}{2}\mathrm{div}(\varphi^2 |A|_\epsilon^2) - \frac{1}{2}\varphi^2 \Delta_{\partial\Omega}(|A|_\epsilon^2) \\
&= \varphi^2 \frac{1}{4|A|_\epsilon^2} |\nabla_{\partial\Omega}|A|^2|^2 + |A|_\epsilon^2 |\nabla_{\partial\Omega}\varphi|^2 + \frac{1}{2}\mathrm{div}(\varphi^2 |A|_\epsilon^2) - \frac{1}{2}\varphi^2 \Delta_{\partial\Omega}(|A|^2).
\end{aligned}$$

Now, $\nabla_{\partial\Omega}(|A|^2)$ and $\Delta_{\partial\Omega}(|A|^2)$ vanish $\mathcal{H}^{n-1}$-almost everywhere on the set $\{|A| = 0\} \cap \partial\Omega$, and, on the set $\{|A| \neq 0\}$, $\frac{1}{4|A|_\epsilon^2} |\nabla_{\partial\Omega}|A|^2|^2 = \frac{|A|^2}{|A|_\epsilon^2} |\nabla_{\partial\Omega}|A||^2$, thus,

$\frac{1}{4|A|_\epsilon^2} |\nabla_{\partial\Omega}|A|^2|^2 \le \mathbb{1}_{|A|\neq 0} |\nabla_{\partial\Omega}|A||^2$. By (1.7) and by the tangential divergence theorem, using that the mean curvature of $\partial\Omega$ vanishes,

$$\int_{\partial\Omega} \varphi^2 \Big(|\nabla_{\partial\Omega}|A||^2 \mathbb{1}_{\{|A|\neq 0\}} - \frac{1}{2}\Delta_{\partial\Omega}(|A|^2)\mathbb{1}_{\{|A|\neq 0\}} - |A|^2 |A|_\epsilon^2 \Big) \ge - \int_{\partial\Omega} |A|_\epsilon^2 |\nabla_{\partial\Omega}\varphi|^2.$$

Letting $\epsilon \to 0$, and using (1.9), we can write

$$\int_{\partial\Omega} \varphi^2 \frac{2|A|^2}{|x|^2} - |A|^2 |\nabla_{\partial\Omega}\varphi|^2 \le 0,$$

and, since $|A| \lesssim \frac{1}{|X|}$, by smoothing and approximation, the previous inequality holds for any φ Lipschitz such that

$$\int\limits_{\partial\Omega} \frac{\varphi^2}{|x|^4} < \infty.$$

Take now $\varphi := u(|x|)$; by the Coarea formula,

$\int\limits_{\partial\Omega} \frac{\varphi^2}{|x|^4} = \mathcal{H}^{n-2}(\partial\Omega \cap \mathcal{S}^{n-1}) \int\limits_0^\infty u^2(r) r^{n-6}\, dr$, where we used that, since $\nu(x) \cdot x = 0$, $\nabla_{\partial\Omega}|x| = \nabla|x|$, and thus the coarea factor of $|x|$ on $\partial\Omega$ is equal to 1. Thus, using also that φ is 0-homogeneus and hence $\nabla_{\partial\Omega}\varphi = \nabla\varphi$, the stability inequality reduces to

$$\int\limits_{\partial\Omega} |A|^2 \left(|u'(|x|)|^2 - 2\frac{u^2(|x|)}{|x|^2} \right) \geq 0,$$

whenever $\int\limits_0^\infty u^2(r) r^{n-6}\, dr < \infty$. Let us define

$$u(r) := \begin{cases} r^\alpha, & 0 < r < 1, \\ r^\beta, & r > 1, \end{cases}$$

where, in order to have u Lipschitz, we impose $\alpha \geq 1$ and $\beta < 0$. The integrability condition on u is the finiteness of $\int\limits_0^1 r^{2\alpha+n-6}$ and $\int\limits_1^\infty r^{2\beta+n-6}$, that is

$$\beta < \frac{5-n}{2} < \alpha.$$

Under this condition, the stability inequality implies

$$(\alpha^2 - 2) \int\limits_{B\cap\partial\Omega} |A|^2 |x|^{2(\alpha-1)} + (\beta^2 - 2) \int\limits_{\partial\Omega\setminus B} |A|^2 |x|^{2(\beta-1)} \geq 0.$$

Now, since $3 \leq n \leq 7$, there exist α and β such that

$$\beta < 0, \qquad \alpha \geq 1, \qquad \beta < \frac{5-n}{2} < \alpha, \qquad \alpha^2 < 2, \qquad \beta^2 < 2,$$

and thus

$$\int\limits_{B\cap\partial\Omega} |A|^2 |x|^{2(\alpha-1)} = 0 = \int\limits_{\partial\Omega\setminus B} |A|^2 |x|^{2(\beta-1)},$$

that implies $|A| \equiv 0$, i.e. ν is constant on $\partial\Omega \setminus \{0\}$, and then Ω is an half-space. $\square$

Notice that the Simons' Theorem, together with Theorem 1.3.7, imply that $n^* \geq 8$.

The critical dimension is indeed 8, as shown by the following Theorem due to Bombieri, De Giorgi and Giusti [6] .

Theorem 1.3.11. *If $m \geq 4$, then the Simons cone*

$$\Omega := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m : |x| < |y|\}$$

is a smooth minimizing cone.

CHAPTER 2

Regularity of capillary hypersurfaces

In sections 1, 2, 3 and 5 of this chapter we will follow the work of Edelen, Chodosh, and Li in [2]. Let $\sigma \in (0,1)$ be a real number, and let $E \subset \mathbb{R}^{n+1}_+$ be a set of finite perimeter. We will work with the functional

$$\mathcal{A}(E) := \mathcal{H}^n(\partial^* E \cap \mathbb{R}^{n+1}_+) - \sigma \mathcal{H}^n(\partial^* E \cap \mathbb{R}^n), \tag{2.1}$$

where $\mathbb{R}^{n+1}_+$ is the upper half space and $\mathbb{R}^n := \partial \mathbb{R}^{n+1}_+$. If E is a set of locally finite perimeter, and U is a bounded open set in $\mathbb{R}^{n+1}$, we will denote

$$\mathcal{A}_U(E) := \mathcal{A}(U \cap E)$$

Like we did in the previous chapter with the perimeter, we want to study the behaviour of minimizers of $\mathcal{A}$ among the class of sets of locally finite perimeter.

Definition 2.0.1. Let A be an open set in $\mathbb{R}^{n+1}$, and let E be a set of locally finite perimeter in $\mathbb{R}^{n+1}_+$. We say that E is a minimizer for $\mathcal{A}$ in A if, for any $U \subset A$ bounded open set, and any $E' \subset \mathbb{R}^{n+1}_+$ of locally finite perimeter such that $E' \Delta E \subset\subset A$, then

$$\mathcal{A}_U(E) \leq \mathcal{A}_U(E').$$

If $A = \mathbb{R}^{n+1}$, we will say that E is a minimizer of $\mathcal{A}$.

Definition 2.0.2. Let E be an open set in $\mathbb{R}^{n+1}_+$, and denote $M := \partial E \cap \mathbb{R}^{n+1}_+$. Let A be an open set in $\mathbb{R}^{n+1}$. We say that E is smooth in A if the clousure of M in A is a smooth hypersurface. We denote as $\partial M \subset \mathbb{R}^n$ the boundary of M, and we call M a capillary surface.

If E is a minimizer of $\mathcal{A}$ smooth in A, we will say that E is a smooth minimizer in A. If $A = \mathbb{R}^{n+1}$, we say that E is a smooth minimizer.

If E is a cone, and $A = \mathbb{R}^{n+1} \setminus \{0\}$, we will say that E is a smooth cone.

2.1 First and second variation

Let E be a smooth minimizer of $\mathcal{A}$. In particular, for every admissible local variation E_t of E,

$$\frac{d}{dt}\mathcal{A}(E_t)_{|t=0} = 0, \qquad\qquad \frac{d^2}{dt^2}\mathcal{A}(E_t)_{|t=0} \geq 0.$$

Since we work in the upper half space we need variations with velocity vector field tangential to $\mathbb{R}^n$. Namely, for $T \in C^\infty_c(\mathbb{R}^{n+1};\mathbb{R}^n)$, we take the variation $\Phi_t(x)$ defined by the ODE

$$\begin{cases} \frac{d}{dt}\Phi_t(x) = T(\Phi_t(x)), \\ \Phi_0(x) = x, \end{cases}$$

and we say that Φ is a variation with initial velocity vector field T.

Definition 2.1.1. Let $E \subset \mathbb{R}^{n+1}_+$ be smooth.

We say that E is stationary for $\mathcal{A}$, if

$$\frac{d}{dt}\mathcal{A}(E_t)_{|t=0} = 0,$$

for any local variation E_t of A.

We say that E is stable for $\mathcal{A}$, if

$$\frac{d^2}{dt^2}\mathcal{A}(E_t)_{|t=0} \geq 0,$$

for any local variation E_t of A.

Denote $M := \partial E \cap \mathbb{R}^{n+1}_+$, and let ν be the unit normal vector field of M in E that points out of E, η be the unit normal vector field of ∂M in M that points out of M, e_{n+1} be the $(n+1)^{th}$ vector of the canonical basis of $\mathbb{R}^{n+1}$, and $\overline{\nu}$ be the unit normal vector field of ∂M in $\mathbb{R}^n$ that points out of $\partial E \cap \mathbb{R}^n$. Notice that, since $e_{n+1} \perp \partial M$ and $\overline{\nu} = \frac{\nu - (\nu\cdot e_{n+1})e_{n+1}}{\sqrt{1-(\nu\cdot e_{n+1})^2}}$, we have

$$\overline{\nu} \in \mathrm{Span}(\nu, \eta), \qquad\qquad e_{n+1} \in \mathrm{Span}(\nu, \eta). \tag{2.2}$$

Take $T \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^n)$ and define $E_t := \Phi_t(E)$. Using the first variation of the area, the tangential divergence theorem, and the area formula, it's easy to see that, for t small enough,

$$\begin{aligned}\frac{d}{dt}\mathcal{A}(E_t) = &\int_M H_t(\Phi_t(x))T(\Phi_t(x)) \cdot \nu_t(\Phi_t(x))\mathrm{J}_M\Phi_t(x)d\,\mathcal{H}^n(x)+ \\ &+ \int_{\partial M} T(\Phi_t(x)) \cdot (\eta_t(\Phi_t(x)) - \sigma\overline{\nu}_t(\Phi_t(x)))\mathrm{J}_{\partial M}\Phi_t(x)\,d\,\mathcal{H}^{n-1}(x),\end{aligned} \tag{2.3}$$

where H_t is the mean curvature of $M_t := \Phi_t(M)$, ν_t is the unit normal vector field of M_t in E_t that points out of E_t, η_t is the unit normal vector field of ∂M_t in M_t that points out of M_t, $\overline{\nu}_t$ is the unit normal vector field of ∂M_t in $\mathbb{R}^n$ that points out of $\partial E_t \cap \mathbb{R}^n$, $\mathrm{J}_M\Phi_t(x)$ is the Jacobian of the isomorphism $d\Phi_t(x) : T_xM \to T_{\Phi_t(x)}M_t$, and $\mathrm{J}_{\partial M}\Phi_t(x)$ is the Jacobian of the isomorphism $d\Phi_t(x) : T_x\partial M \to T_{\Phi_t(x)}\partial M_t$.
From (2.3) and the stationarity of E we get

$$H_0 = 0, \qquad\qquad \eta - (\eta \cdot e_{n+1})e_{n+1} = \sigma\overline{\nu}, \tag{2.4}$$

from which, taking into account that $\eta \perp \nu$, and that $\overline{\nu} \in \mathrm{Span}(\nu, \eta)$,

$$\cos(\theta) = \sigma, \tag{2.5}$$

where $\cos(\theta) := \nu \cdot e_{n+1}$. Now we compute the second variation $\frac{d^2}{dt^2}\mathcal{A}(E_t)_{|t=0}$ taking the derivative of (2.3). Derivating under the integral sign and using (2.4), we obtain

$$\frac{d^2}{dt^2}\mathcal{A}(E_t)_{|t=0} = \int_M \frac{d}{dt}(H_t\circ\Phi_t)_{|t=0} T\cdot\nu\, d\mathcal{H}^n + \int_{\partial M} T\cdot\frac{d}{dt}(\eta_t\circ\Phi_t - \sigma\overline{\nu}_t\circ\Phi_t)_{|t=0}\, d\mathcal{H}^{n-1}, \tag{2.6}$$

therefore we need to compute $\frac{d}{dt}(H_t \circ \Phi_t)_{|t=0}$ and $\frac{d}{dt}(\eta_t - \sigma\overline{\nu}_t)_{|t=0}$. We will deal first with the derivative of the unit normal vector fields ν_t, η_t, and $\overline{\nu}_t$.
We recall that, since $\tilde{E}$ is an open set with smooth boundary, there exists a smooth real-valued map $\tilde{s}$ defined on a neighbourhood of $\partial\tilde{E}$, such that

$N := \nabla\tilde{s}$ is an extension of ν with $|N| \equiv 1$. Therefore $A := \nabla^2\tilde{s}$ is a symmetric extension of the second fundamental form of $\partial\tilde{E}$ satisfying $A\,N = 0$. Rotating N of $\frac{\pi}{2}$ in the plane defined by N and e_{n+1}, we obtain a smooth extension Z of η to a neighbourhood of ∂M, satisfying $|Z| \equiv 1$ and $Z \perp N$. Therefore,

$$(\nabla Z)^t Z = 0, \qquad (\nabla Z)^t N = -(\nabla N) Z. \tag{2.7}$$

Finally, we give an extension $\overline{N}$ of $\overline{\nu}$ defined by $\overline{N} = \frac{N-(N\cdot e_{n+1})e_{n+1}}{\sqrt{1-(N\cdot e_{n+1})^2}}$.
By a simple computation we can express ν_t, η_t and $\overline{\nu}_t$ in terms of ν, η and $\overline{\nu}$:

$$\nu_t \circ \Phi_t = \frac{(\nabla\Phi_t)^{-t}\nu}{|(\nabla\Phi_t)^{-t}\nu|}, \tag{2.8}$$

$$\eta_t \circ \Phi_t = \frac{(\nabla\Phi_t)^{-t}\eta \,-\, (\nu_t\circ\Phi_t \cdot (\nabla\Phi_t)^{-t}\eta)\nu_t\circ\Phi_t}{|(\nabla\Phi_t)^{-t}\eta \,-\, (\nu_t\circ\Phi_t \cdot (\nabla\Phi_t)^{-t}\eta)\nu_t\circ\Phi_t|}, \tag{2.9}$$

$$\overline{\nu}_t \circ \Phi_t = \frac{(\nabla\Phi_t)^{-t}\overline{\nu} \,-\, (e_{n+1} \cdot (\nabla\Phi_t)^{-t}\overline{\nu})e_{n+1}}{|(\nabla\Phi_t)^{-t}\overline{\nu} \,-\, (e_{n+1} \cdot (\nabla\Phi_t)^{-t}\overline{\nu})e_{n+1}|}. \tag{2.10}$$

Using these relations and the fact that $\nu\cdot\eta = 0, \overline{\nu}\cdot e_{n+1} = 0$, and that $T\cdot e_{n+1} = 0$ provides $(\nabla T)^t e_{n+1} = 0$, we can compute

$$\frac{d}{dt}\nu_t \circ \Phi_t\,_{|t=0} = -(\nabla T)^t\nu + (\nu\cdot(\nabla T)^t\nu)\nu, \tag{2.11}$$

$$\frac{d}{dt}\eta_t \circ \Phi_t\,_{|t=0} = -(\nabla T)^t\eta + (\nu\cdot(\nabla T)^t\eta)\nu + (\eta\cdot(\nabla T)^t\nu)\nu + (\eta\cdot(\nabla T)^t\eta)\eta, \tag{2.12}$$

$$\frac{d}{dt}\overline{\nu}_t \circ \Phi_t\,_{|t=0} = -(\nabla T)^t\overline{\nu} + (e_{n+1}\cdot(\nabla T)^t\overline{\nu})e_{n+1} + (\overline{\nu}\cdot(\nabla T)^t\overline{\nu})\overline{\nu}. \tag{2.13}$$

Taking from now on

$$T = \frac{\varphi}{\sqrt{1-(N\cdot e_{n+1})^2}}\overline{N}, \tag{2.14}$$

with $\varphi \in C_c^\infty(\partial M)$, we have $T \in \mathrm{Span}(N, Z)$, therefore $T = (T\cdot N)N + (T\cdot Z)Z$. Since on ∂M we have $T = (T\cdot\overline{\nu})\overline{\nu} = (T\cdot\eta)\eta + (T\cdot\nu)\nu$, we have

$$T\cdot\left(\frac{d}{dt}\overline{\nu}_t \circ \Phi_t\,_{|t=0}\right) = -(T\cdot\overline{\nu})\overline{\nu}\cdot(\nabla T)^t\overline{\nu} + (\overline{\nu}\cdot(\nabla T)^t\overline{\nu})(T\cdot\overline{\nu}) = 0, \tag{2.15}$$

and

$$T\cdot\left(\frac{d}{dt}\eta_t\circ\Phi_{t\;|t=0}\right) = -T\cdot(\nabla T)^t\eta+\big((T\cdot\nu)\nu\cdot(\nabla T)^t\eta)\big)+\big((T\cdot\eta)\eta\cdot(\nabla T)^t\eta)\big)+$$
$$+(T\cdot\nu)(\eta\cdot(\nabla T)^t\nu) = (T\cdot\nu)(\eta\cdot(\nabla T)^t\nu). \quad (2.16)$$

Using now $\nabla T = (T\cdot N)A + (T\cdot Z)\nabla Z + N\otimes\nabla(T\cdot N) + Z\otimes\nabla(T\cdot Z)$, where by $v\otimes w$ we mean $v\cdot w^t$, and taking into account that $A\,N = 0$, we have

$$(T\cdot\nu)(\eta\cdot(\nabla T)^t\nu) = (T\cdot N)(Z\cdot(\nabla T)^t N) = (T\cdot N)(T\cdot Z)(Z\cdot(\nabla Z)^t N)+(T\cdot N)\nabla(T\cdot N)\cdot Z.$$

Now we use that $(\nabla Z)^t N = -A\,Z$, that $T\cdot N = \varphi$, and that, on ∂M,

$$T\cdot\eta = -\varphi\frac{\cos(\theta)\cos(\pi/2+\theta)}{\sin^2(\theta)} = \varphi\cot(\theta),$$

in order to deduce, from (2.15) and (2.16),

$$\int_{\partial M} T\cdot\frac{d}{dt}(\eta_t\circ\Phi_t-\sigma\overline{\nu}_t\circ\Phi_t)_{|t=0}\,d\mathcal{H}^{n-1} = \int_{\partial M} -\cot(\theta)\varphi^2\eta\cdot A\eta+\varphi\nabla(\varphi)\cdot\eta\,d\mathcal{H}^{n-1}. \quad (2.17)$$

We need now to compute the derivative of the mean curvature.
Notice that N provides an extension N_t of ν_t defined by $N_t\circ\Phi_t = \frac{(\nabla\Phi_t)^{-t}N}{|(\nabla\Phi_t)^{-t}N|}$, with $|N_t|\equiv 1$. Therefore we have $(\nabla N_t)^t N_t = 0$, which implies $H_t\circ\Phi_t = tr(\nabla(N_t)\circ\Phi_t) = tr((\nabla\Phi_t)^{-t}\nabla(N_t\circ\Phi_t))$. Then,

$$\frac{d}{dt}H_t\circ\Phi_{t\;|t=0} = -tr((\nabla T)^t A) + \operatorname{div}\big(-(\nabla T)^t N + (N\cdot(\nabla T)^t N)N\big) =$$
$$= -tr((\nabla T)^t A) + \operatorname{div}\left(-\nabla_M(T\cdot N) + A\,T\right),$$

where $-\nabla_M$ is the gradient tangential to M. Using that the divergence of a tangential vector field is equal to its tangential divergence div_M, we have $\operatorname{div}(\nabla_M(T\cdot N)) = \operatorname{div}_M(\nabla_M(T\cdot N))$. Notice that, if $x\in M$, $Z(x)\in T_xM$, therefore, if f is a smooth function which is zero on M, $\nabla f\cdot Z = 0$ on M. Taking $f = \Delta\tilde{s}$, which is equal to $H_0 = 0$ on M, and taking into account that $A = \nabla^2\tilde{s}$ and that $A\,N = 0$, we have

$$\operatorname{div}(A\,T) = \partial_i\left(\partial_i\partial_j\tilde{s}\,(T\cdot Z)Z_j\right) = (T\cdot Z)\nabla(\Delta\tilde{s})\cdot Z+tr(\nabla((T\cdot Z)\,Z)^t A) = tr(\nabla((T\cdot Z)\,Z)^t A),$$

where we used the Einstein summation convention for repeated indices.
Since $-tr((\nabla T)^t A) = -tr(\nabla((T\cdot Z)\,Z)^t A) - (T\cdot N)|A|^2$,

$$\frac{d}{dt}H_t\circ\Phi_{t\;|t=0} = -\operatorname{div}_M\nabla_M(\varphi) - \varphi|A^2|, \quad (2.18)$$

which gives, together with the tangential divergence theorem, and with $H_0 \equiv 0$ on M,

$$\int_M \frac{d}{dt} H_t \circ \Phi_{t\,|t=0} (T \cdot \nu)\, d\mathcal{H}^n = \int_M |\nabla_M \varphi|^2 - |A|^2 \varphi^2 \, d\mathcal{H}^n - \int_{\partial M} \varphi \nabla_M \varphi \cdot \eta \, d\mathcal{H}^{n-1}. \tag{2.19}$$

Now, η is tangential to M at ∂M, then $\nabla_M \varphi \cdot \eta = \nabla \varphi \cdot \eta$. Combining this relation with (2.19), (2.17), and the minimality of E, we finally obtain the following theorem.

Theorem 2.1.2 (Stability inequality)**.** *Let $E \subset \mathbb{R}^{n+1}_+$ be smooth and stationary for $\mathcal{A}$. Then,*

$$H_M = 0, \qquad \cos(\theta) = \sigma, \tag{2.20}$$

where $\cos(\theta) := e_{n+1} \cdot \nu$. *If, moreover, E is stable for $\mathcal{A}$, it satisfies the following stability inequality*

$$\int_M |\nabla_M \varphi|^2 - |A|^2 \varphi^2 \, d\mathcal{H}^n - \cot(\theta) \int_{\partial M} \varphi^2 \eta \cdot A \eta \, d\mathcal{H}^{n-1} \geq 0, \qquad \text{for all}\, \varphi \in C^\infty_c(M). \tag{2.21}$$

Notice that we proved (2.21) only for $\varphi \in C^\infty_c(\partial M)$, but using a partition of unity and the fact that if φ has support away from the boundary we are left with the well known second variation of the perimeter, we can write (2.21) for every $\varphi \in C^\infty_c(M)$.

Remark 2.1.3. We will apply the previous theorem to a minimizing cone with an isolated singularity at the origin. In this case we are allowed to use only test functions in $C^\infty_c(M \setminus \{0\})$.

Notation 2.1.4. Since, for a stationary smnooth set of $\mathcal{A}$ we have $\cos(\theta) = \sigma$, we will call $\sigma = \cos(\theta)$, and we will call $\mathcal{A} = \mathcal{A}^\theta$, underlyining the dependence on the angle θ.

Remark 2.1.5. Since any smooth minimizer Ω of $\mathcal{A}^\theta$, gives, by complementation, a smooth minimizer $\mathbb{R}^{n+1}_+ \setminus \Omega$ of $\mathcal{A}^{\pi-\theta}$, we will always assume, without loss of generality, that $\theta \in (0, \frac{\pi}{2})$.

Now we prove that, in the special case of a minimizing cone Ω with an isolated singularity at the origin, we can lower the degrees of freedom of the problem in (2.21).
For such an Ω we have that $\Sigma := M \cap S^n$ is smooth, $M = \{\lambda \Sigma \,|\, \lambda > 0\}$, and

all the normal vector fields ν, $\overline{\nu}$, η are 0-homogeneus. Moreover, if $x \in \Sigma$, $\nu(x)$ is the unit normal vector field of Σ in S^n that points out of $\Omega \cap S^n$, $\eta(x)$ is the unit normal vector field of $\partial\Sigma$ is Σ that points out of Σ, and $\overline{\nu}$ is the unit normal vector field of $\partial\Sigma$ in $\partial\Omega \cap S^n \cap \mathbb{R}^n$ that points out of $\partial\Omega \cap S^n \cap \mathbb{R}^n$.

Theorem 2.1.6. *Suppose $\Omega \subset \mathbb{R}^{n+1}_+$ is a cone, stationary and stable for $\mathcal{A}$, with an isolated singularity at the origin. Then we have that*

$$\int_\Sigma (|\nabla_\Sigma f|^2 - |A|^2 f^2)\, d\mathcal{H}^{n-1} - \cot\theta \int_{\partial\Sigma} \eta \cdot A\eta f^2\, d\mathcal{H}^{n-2} \geq -\left(\frac{n-2}{2}\right)^2 \int_\Sigma f^2\, d\mathcal{H}^{n-1}, \tag{2.22}$$

for all $f \in C^1(S^n)$

Proof. We use as a test function in (2.21) $\varphi(\omega r) = g(r) f(\omega)$, where $\omega \in S^n$, $r > 0$, $f \in C^1(S^n)$ and $g \in C_c^\infty(0,\infty)$. Using the coarea formula and that, since ν and f are 0-homogeneus, A and ∇f are $(-1)-$homogeneus, we get

$$\int_0^\infty r^{n-1} \int_\Sigma g'(r)^2 f^2(\omega) + g^2(r) r^{-2} |\nabla_\Sigma f_\omega|^2 - |A_\omega|^2 r^{-2} g^2(r) f^2(\omega)\, \mathcal{H}^{n-1}(\omega)\, dr -$$

$$- \cot\theta \int_0^\infty r^{n-3} g^2(r) \int_{\partial\Sigma} \eta(\omega) \cdot A_\omega \eta(\omega) f^2(\omega)\, d\mathcal{H}^{n-2}(\omega)\, dr \geq 0,$$

and by the Hardy inequality (see Lemma 2.1.7) follows that

$$\frac{(n-2)^2}{4} = \inf\left\{ \frac{\int_0^\infty g'(r)^2\, r^{n-1} dr}{\int_0^\infty g(r)^2 r^{n-3}\, dr} \ : \ g \in C_c^\infty((0,+\infty)) \right\}.$$

□

Lemma 2.1.7. *We have that*

$$\frac{(n-2)^2}{4} = \inf\left\{ \frac{\int_0^\infty g'(r)^2\, r^{n-1} dr}{\int_0^\infty g(r)^2 r^{n-3}\, dr} \ : \ g \in C_c^\infty((0,+\infty)) \right\}.$$

Proof. Let $g \in C_c^\infty((0,+\infty))$. Since $g'(r) r^{\frac{n-1}{2}} = (g(r) r^{\frac{n-1}{2}})' - \frac{n-1}{2} g(r) r^{\frac{n-3}{2}}$, we have that

$$\int_0^{+\infty} g'(r)^2 r^{n-1} dr = \int_0^{+\infty} |(g(r) r^{\frac{n-1}{2}})'|^2\, dr$$

$$- 2\int_0^{+\infty} (g(r)r^{\frac{n-1}{2}})' \frac{n-1}{2} g(r) r^{\frac{n-3}{2}}\, dr$$
$$+ \int_0^{+\infty} (\frac{n-1}{2} g(r) r^{\frac{n-3}{2}})^2(r)\, dr.$$

Now we notice that for every $\varphi \in C_c^\infty((0,+\infty))$ we have

$$\begin{aligned}\int_0^{+\infty} (\varphi(r)r)'\varphi(r)\, dr &= -\int_0^{+\infty} \varphi'(r) r\varphi(r)\, dr \\ &= -\int_0^{+\infty} \Big((\varphi r)' - \varphi\Big)\varphi\, dr \\ &= -\int_0^{+\infty} (\varphi r)'\varphi\, dr + \int_0^{+\infty} \varphi^2\, dr,\end{aligned}$$

which gives that

$$2\int_0^{+\infty} (r\varphi)'\varphi\, dr = \int_0^{+\infty} \varphi^2\, dr,$$

and so

$$2\int_0^{+\infty} (g(r)r^{\frac{n-1}{2}})' g(r) r^{\frac{n-3}{2}}\, dr = \int_0^{+\infty} \Big(g(r) r^{\frac{n-3}{2}}\Big)^2\, dr$$

$$\begin{aligned}\int_0^{+\infty} g'(r)^2 r^{n-1} dr &= \int_0^{+\infty} \left|\Big(g(r) r^{\frac{n-1}{2}}\Big)'\right|^2 dr + \Big(-\frac{n-1}{2} + \Big(\frac{n-1}{2}\Big)^2\Big)\int_0^{+\infty} g(r)^2 r^{n-3}\, dr \\ &= \int_0^{+\infty} \left|\Big(g(r) r^{\frac{n-1}{2}}\Big)'\right|^2 dr + \frac{(n-1)(n-3)}{4}\int_0^{+\infty} g(r)^2 r^{n-3}\, dr.\end{aligned}$$

Setting $\varphi(r) = r^{\frac{n-1}{2}} g(r)$, we get that

$$\frac{\int_0^\infty g'(r)^2\, r^{n-1} dr}{\int_0^\infty g(r)^2 r^{n-3}\, dr} = \frac{\int_0^\infty \big(\varphi'(r)\big)^2\, dr}{\int_0^\infty \frac{1}{r^2}\varphi^2(r)\, dr} + \frac{(d-1)(d-3)}{4}\ .$$

Finally, by the Hardy inequality (see [4]), we have that

$$\frac{1}{4} = \inf\left\{ \frac{\int_0^\infty \big(\varphi'(r)\big)^2\, dr}{\int_0^\infty \frac{1}{r^2}\varphi^2(r)\, dr}\ :\ \varphi \in C_c^\infty((0,+\infty))\right\}.$$

□

2.2 The main Theorem

Like we did for perimeter minimizers, we want to study the regularity of minimizers for $\mathcal{A}^\theta$.

Also in this context, we have regularity away from a singular set, as stated in the following theorem (see [7]).

Theorem 2.2.1. *Let $\theta \in (0, \frac{\pi}{2})$. Let $n \leq 6$, $E \subset \mathbb{R}^{n+1}_+$ be a minimizer of $\mathcal{A}^\theta$, and let $M := \partial E \cap \mathbb{R}^{n+1}_+$. Then, $\overline{M}$ is a smooth hypersurface away from a closed set $\Sigma(M) \subset \overline{M} \cap \mathbb{R}^n$, that we call the singular set of M. Moreover, $\mathcal{H}^{n-1}(\Sigma(M)) = 0$.*

Moreover, it can be proven that the Federer's dimension reduction argument applies also here, so that we have the following Theorem.

Theorem 2.2.2. *Let $\theta \in (0, \frac{\pi}{2})$. There is a critical dimension $n^*(\theta)$ (defined as the first dimension n such that there is a smooth minimizing cone for $\mathcal{A}^\theta$ in $\mathbb{R}^{n+1}$), such that*

- *if $n < n^*$, then the singular set $\Sigma(M)$ is empty for any minimizer of $\mathcal{A}^\theta$ in $\mathbb{R}^{n+1}$;*
- *if $n = n^*$, then for any minimizer of $\mathcal{A}^\theta$ in $\mathbb{R}^{n+1}$, the singular set $\Sigma(M)$ is a discrete set of points;*
- *if $n > n^*$, then for any minimizer of $\mathcal{A}^\theta$ in $\mathbb{R}^{n+1}$, the Hausdorff dimension of $\Sigma(M)$ is at most $n - n^*$, that is:*

$$\mathcal{H}^{n-n^*+\varepsilon}(\Sigma(M)) = 0 \quad \text{for every} \quad \varepsilon > 0.$$

Therefore, we study smooth cones also in this context, proving the following Theorem.

Theorem 2.2.3. *There are angles θ_0 and θ_1 such that the following holds.*

- *$n^*(\theta) \geq 4$, for any $\theta \in (0, \frac{\pi}{2})$.*
- *If $0 < \theta < \theta_0$, then $n^*(\theta) \geq 5$.*
- *If $\theta_1 < \theta < \frac{\pi}{2}$, then $n^*(\theta) \geq 7$.*

2.3 The case $n = 3$

We are going to prove that there are no minimizing cones (for (2.1)) with an isolated singularity in $\mathbb{R}^4_+$. In this case Σ is a surface embedded in S^3. We will use (2.22) combined with the Gauss-Bonnet Theorem to find out that $\chi(C_\Sigma) > 0$, for every connected component of Σ, where χ is the Euler characteristic. By the classification of compact oriented connected surfaces with boundary will folllow that Σ is homeomorphic to a disk. By the uniformization theorem (see [8]), then, will follow that Σ is conformally equivalent to a disk, and we will use a conformal parametrization of Σ to show that $|A|^2 \equiv 0$ on Σ, and by homogeneity $|A|^2 \equiv 0$ on M. This, together with the boundary contact angle condition, implies that Ω is the intersection of an half-space with $\mathbb{R}^{n+1}_+$, therefore it is smooth also in the origin.
We need some Lemmas and definitions before the proof.
Notice that the second fundamental form A of a cone satisfies $A_x\, x = A_x\, \nu(x) = 0$, then A can be viewed as a symmetric tensor field on $T\Sigma$. The second fundamental form of Σ, as a submanifold of S^3, is defined through

$$\sigma(v, w) = \nabla_v \nu \cdot w, \tag{2.23}$$

where ∇ is the Levi-Civita connection on S^3 induced by the euclidean metric, and $v, w \in T\Sigma$. Since, in order to use the Gauss-Bonnet theorem, we will need σ to compute the geodesic curvature of $\partial\Sigma$, we now exploit the relation between A and σ.

Lemma 2.3.1. *For every $v, w \in T_x\Sigma$, and for every $Y, \tilde{Y}$ vector fields on Σ,*

$$v \cdot A_x\, w = \sigma_x(v, w), \qquad Y(p) \cdot A_p\, \tilde{Y}(p) = -\nabla_Y \tilde{Y}(p) \cdot \nu(p) \tag{2.24}$$

Proof. For a fixed $x \in T_x\Sigma$, we can choose a coordinate system for Σ such that the metric is 0 at the first order in x. Therefore, $\nabla_v \nu(x) = \partial_v \nu(x) = A_x\, v$, that gives the first equality in (2.24), which is independent on the coordinate system.
Differentiating $\tilde{Y} \cdot \nu = 0$ we obtain

$$\nabla_Y \nu \cdot \tilde{Y} = -\nabla_Y \tilde{Y} \cdot \nu,$$

that ends the proof. □

For a more detailed version of the following Lemma we refer to [9].

Lemma 2.3.2. *Let Σ as above. Then, at $\partial\Sigma$,*

$$\cot\theta \eta \cdot A\eta = -k_g, \tag{2.25}$$

where k_g is the geodesic curvature of $\partial\Sigma$.

Proof. By definition, $k_g = -\nabla_\tau \tau \cdot \eta$. Since M has zero mean curvature, $\eta \cdot A\eta = -\tau A \cdot \tau$, where τ is the unit tangent vector field of $\partial\Sigma$. Therefore, by the previous Lemma,

$$\cos\theta\, \eta \cdot A\eta + \sin\theta\, k_g = -\nabla_\tau \tau \cdot (-\cos\theta\, \nu + \sin\theta\, \eta).$$

Now, $-\cos\theta\, \nu + \sin\theta\, \eta = -e_{n+1}$, and since τ is orthogonal to e_{n+1}, so is $\nabla_\tau \tau$. □

Definition 2.3.3. Let $\mathbb{N}$ be a Riemannian manifold with dimension 2. We define the Gaussian curvature K of N by

$$K = \frac{R_N}{2}, \tag{2.26}$$

where R_N is the Ricci curvature of N.

Remark 2.3.4. If (M, g) is a Riemannian manifold and x^μ are coordinates on M, the Christoffel symbol with respect to the Levi-Civita connection are given by

$$\Gamma^\lambda_{\mu,\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \tag{2.27}$$

where we used the Einstein convention for repeated indexes, and where $g^{\lambda\sigma}$ are the coordinates of g^{-1}. We recall that the Ricci curvature at a point $p \in M$ is given by

$$R_M := \partial_\mu \Gamma^\mu_{\nu\nu} - \partial_\nu \Gamma^\mu_{\mu\nu} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{\nu\nu} - \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\mu\nu}, \tag{2.28}$$

whenever $g_{\mu\nu}(p) = \delta_{\mu\nu}$, and we will compute the Ricci curvature at a point only in those particular coordinate systems.

Lemma 2.3.5. *Let M as above. Then,*

$$R_M = -|A|^2. \tag{2.29}$$

Proof. Notice that (2.29) doesn't depend on the coordinate system. Therefore we can fix $p \in M$, and, up to translation and rotations, we can assume that $p = 0$, and that, locally in 0, M is the graph of a function $u(x) : \mathbb{R}^n \to \mathbb{R}$, such that $u(0) = 0$, $\nabla u(0) = 0$, and $D^2 u(0)$ is diagonal. In this coordinate system

$$g = Id + \nabla u \otimes \nabla u.$$

Therefore, using that, for every matrix A with $\|A\| < 1$, the Neumann series gives

$$(Id + A)^{-1} = \sum_{k=0}^{\infty} (-1)^k A^k,$$

the Taylor expansion of ∇u in 0 gives

$$g^{\mu\nu}(x) = \delta^{\mu\nu} + O(|x|^2).$$

Moreover, the external normal to M in Ω is

$$\nu \circ u(x) = (\nabla u, -1)|1 + |\nabla u|^2|^{-1/2},$$

which implies

$$A(0) = D^2u(0), \tag{2.30}$$

and, since the mean curvature of M is zero,

$$\Delta u(0) = 0. \tag{2.31}$$

Computing the Christoffel symbols,

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}g^{\lambda\sigma}(\partial_\mu(u_\nu u_\sigma) + \partial_\nu(u_\sigma u_\mu) - \partial_\sigma(u_\mu u_\nu)) = \\ &= g^{\lambda\sigma}u_\sigma u_{\mu\nu} = \\ &= u_\lambda u_{\mu\nu} + O(|x|^2),\end{aligned}$$

in particular $\Gamma^\lambda_{\mu\nu}(0) = 0$. Therefore the Ricci curvature in 0 is

$$\begin{aligned}R_M(0) &= \partial_\mu \Gamma^\mu_{\nu\nu}(0) - \partial_\nu \Gamma^\mu_{\mu\nu}(0) = \\ &= \partial_\mu(u_\mu u_{\nu\nu})(0) - \partial_\nu(u_\mu u_{\nu\mu})(0) = \\ &= (\Delta u(0))^2 - |D^2u(0)|^2 = \\ &= -|A|^2(0).\end{aligned}$$

□

Lemma 2.3.6. *Let Σ and M as above(For this Lemma we don't require stationarity or stability for M). Then,*

$$K_\Sigma = \frac{R_M}{2} + 1 \tag{2.32}$$

Proof. Let us denote as $\tilde{\Gamma}^\lambda_{\mu\nu}$ the Christoffel symbols of Σ, and as $\tilde{g}$ the metric on Σ. Take a point $p \in \Sigma$ and let $\tilde{\varphi} : U \subset \mathbb{R}^2 \to \Sigma$ be a local parametrization of Σ in $p = \tilde{\varphi}(0)$ such that, in the coordinates induced by $\tilde{\varphi}$, $\tilde{g}(0)_{\mu\nu} = \delta_{\mu\nu}$, and $\tilde{\Gamma}^\lambda_{\mu\nu}(0) = 0$. Notice that the function $\varphi : U \times \mathbb{R}_+ \to M$ defined by $\varphi(x, r) = r\tilde{\varphi}(x)$ is a local parametrization of M around p that agrees on

$U \times \{1\}$ with $\tilde{\varphi}$. Let us work in the coordinates induced by φ on M. Notice that

$$\begin{cases} g_{\mu\nu}(x,r) = r\tilde{g}_{\mu\nu}(x) & \text{if } \mu,\nu \le 2, \\ g_{\mu 3}(x,r) = r\tilde{\varphi}_\mu(x) \cdot \tilde{\varphi} = 0 & \text{if } \mu \le 2, \\ g_{33}(x,r) = |\tilde{\varphi}|^2(x) = 1, \end{cases}$$

where we used that $\Sigma \subset S^3$ and that $\tilde{\varphi} \cdot \tilde{\varphi} = 1$ implies $\tilde{\varphi}_\mu \cdot \tilde{\varphi} = 0$. Therefore the Christoffel symbols satisy

$$\begin{cases} \Gamma^\lambda_{\mu\nu}(x,r) = \tilde{\Gamma}^\lambda_{\mu\nu}(x) & \text{if } \mu,\nu,\lambda \le 2 \\ \Gamma^3_{\mu\nu}(x,r) = -r\tilde{g}_{\mu\nu}(x) = -r + o(|x|) & \text{if } \mu,\nu \le 2 \\ \Gamma^3_{\mu 3} = \Gamma^3_{3\mu} = 0 \\ \Gamma^\lambda_{3\nu}(x,r) = r^{-1}\delta^\lambda_\nu & \text{if } \lambda,\nu \le 2 \\ \Gamma^\lambda_{33} = 0 & \text{if } \lambda \le 2. \end{cases}$$

In particular, the Christoffel symbols of Σ are the same in the coordinates induced by φ and the ones induced by $\tilde{\varphi}$, and every time that two indexes between λ, μ, ν are equal to 3 we have $\Gamma^\lambda_{\mu\nu} = 0$. Therefore, using also $\tilde{\Gamma}^\lambda_{\mu\nu}(0) = 0$ and $\tilde{g}_{\mu\nu}(0) = \delta_{\mu\nu}$,

$$\begin{aligned} R_M(0,1) &= R_\Sigma(0) + (\partial_3 \Gamma^3_{\nu\nu} - \partial_3 \Gamma^\mu_{\mu 3} + \Gamma^\mu_{\mu 3}\Gamma^3_{\nu\nu} - \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\mu\nu})(0,1) = \\ &= R_\Sigma(0) - \tilde{g}_{\nu\nu}(0) + 2 - 2\tilde{g}_{\nu\nu}(0) - (-2\tilde{g}_{\nu\lambda}(0)\delta^\lambda_\nu + 2) = \\ &= R_\Sigma(0) - 2 + 2 - 4 - (-4 + 2) = \\ &= R_\Sigma(0) - 2. \end{aligned}$$

Then

$$K_\Sigma(0) = \frac{R_\Sigma(0)}{2} = \frac{R_M(0,1)}{2} + 1,$$

and the fact that $\varphi(0,1) = \tilde{\varphi}(0) = p$ ends the proof. □

Definition 2.3.7. A diffeomorphism $u : (N, g) \to (\tilde{N}, \tilde{g})$ between Riemannian manifolds is conformal if there exists a smooth function $\lambda \ge 0$ on N such that

$$\tilde{g}\left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\right) = \lambda(x) g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right),$$

for every local parametrization x of N.

Lemma 2.3.8. *Let Σ and Ω as above, D be the closed unit disk in the complex plane. If $u : D \simeq \Sigma$ is a conformal diffeomorphism, the following relations hold*

$$\nu_i \cdot u_j = A_{ij} = -\nu \cdot u_{ij} \tag{2.33}$$

$$\Delta u \cdot \nu \equiv 0 \tag{2.34}$$

$$u_{1k} \cdot u_2 = -u_{2k} \cdot u_1 \tag{2.35}$$

$$u_{ij} \cdot u_i = u_{kj} \cdot u_k \tag{2.36}$$

$$\nu_1 \cdot u_{22} = \nu_2 \cdot u_{12} \tag{2.37}$$

$$\nu_2 \cdot u_{11} = \nu_1 \cdot u_{12} \tag{2.38}$$

where $\frac{A_{i,j}}{\lambda} = \frac{u_i A u_j}{\lambda}$ are the coordinates of the second fundamental form under the parametrization u, and ν_i is the $i^t h-$partial derivative of $\nu \circ u$. Here $\lambda = |u_i|^2 = |u_j|^2$.

Proof. u_i is tangent to Σ, therefore, taking the derivative of $u_i \cdot (\nu \circ u) = 0$, we obtain (2.33). Since M has zero mean curvature, and since u_1, u_2 gives an orthogonal basis of $T\Sigma$, we have also (2.34). (2.35) follows by taking the $k^{th}-$derivative of the relation $u_1 \cdot u_2 = 0$,and aking the $j^t h-$derivative of $u_i \cdot u_i = u_k \cdot u_k$ gives (2.36). From $\nu \cdot u = 0$ and $\nu \cdot \nu = 1$ we have

$$\begin{aligned} \nu_i \cdot u = -\nu \cdot u_i = 0, \\ \nu_i \cdot \nu = 0, \end{aligned}$$

which means $\nu_i \in T\Sigma$. Therefore, using the previous equations,

$$\begin{aligned} \lambda \nu_1 \cdot u_{22} &= (\nu_1 \cdot u_1) u_1 \cdot u_{22} + (\nu_1 \cdot u_2) u_2 \cdot u_{22} = \\ &= (-\nu_2 \cdot u_2)(-u_2 \cdot u_{12}) + (\nu_2 \cdot u_1)(u_1 \cdot u_{12}) = \\ &= \lambda \nu_2 \cdot u_{12}. \end{aligned}$$

(2.38) holds with the same proof. □

Lemma 2.3.9. *Let Ω and Σ as above. Then, for every $x \in \partial\Sigma$,*

$$A_x \, \eta(x) \perp \partial\Sigma. \tag{2.39}$$

Proof. Let $x \in \partial\Sigma$ and $\tau(x) \in T_x\partial\Sigma$. Since $n = 3$, $(x, \nu(x), \eta(x), \tau(x))$ is an orthogonal basis of $\mathbb{R}^4$. Since $\nabla N_x \, x = \nabla N_x \, \nu(x) = 0$, the image of ∇N_x lays in Span$(\eta(x), \tau(x))$. Let γ be a curve in $\partial\Sigma$ such that $\gamma(0) = x$ and $\gamma'(0) = \tau(x)$. From $(\nu \circ \gamma) \cdot e_{n+1} = \cos\theta$ we have

$$e_{n+1} \cdot \nabla N \, \tau = 0.$$

Now, $\nabla N\, \tau = \mu_1 \eta + \mu_2 \tau$, and, since at the boundary $\eta \cdot e_{n+1} = \sin\theta \neq 0$, and $\tau \cdot e_{n+1} = 0$, we have $\mu_1 = 0$, so that

$$\eta \cdot \nabla N\, \tau = 0.$$

□

Theorem 2.3.10. *Let M and Σ as above. Then $|A|^2 \equiv 0$*

Proof. Up to replace Σ with one of its connected components we can assume Σ connected.
Apply (2.22) with $f \equiv 1$, and using that, by previous lemmas, $-\frac{|A|^2}{2} = K_\Sigma - 1$, we get

$$\int_\Sigma K_\Sigma \, d\mathcal{H}^2 + \int_{\partial\Sigma} k_g \, d\mathcal{H}^1 \geq \mathcal{H}^2(\Sigma) \left(\frac{-1}{4} + 1 \right) + \int_\Sigma \frac{|A|^2}{2} \, d\mathcal{H}^2.$$

By Gauss-Bonnet theorem and $\mathcal{H}^2(\Sigma) > 0$ we have that $\chi(\Sigma) > 0$, and as previously discussed, we can find a conformal diffeomorphism u between the complex unit disc D and Σ.
Take complex coordinates $z = x + iy$ and polar coordinates $z = re^{i\alpha}$ on D. since u is conformal, $\frac{\partial u}{\partial r} \perp \frac{\partial u}{\partial \alpha}$. Since at the boundary $\frac{\partial u}{\partial \alpha} \in T\partial\Sigma$, $\frac{\partial u}{\partial r}$ must be proportional to η, and, by Lemma 2.3.9, at the boundary holds

$$A_{r,\alpha} := \frac{\partial u}{\partial r} \cdot A \frac{\partial u}{\partial \alpha} = 0 \qquad \text{on} \qquad \partial D.$$

On the other hand, on the whole disk we have

$$rA_{r,\alpha} = xy(A_{22} - A_{11}) + A_{12}(x^2 - y^2) = \Im \left(\frac{z^2}{2}(A_{22} - A_{11} + 2iA_{12}) \right).$$

Define then

$$2h(z) := A_{22} - A_{11} + 2iA_{12} = 2A_{22} + 2iA_{12}.$$

If we prove that h is holomorphic, so is $z^2 h$. But $z^2 h$ has immaginary part zero at ∂D, therefore the harmonic function $\Im(z^2 h(z))$ is identically zero by the maximum principle. From this, we have that $z^2 h(z)$ is an holomorphic functions with real values that is zero in zero, so it is identically zero. Therefore so is h, which means $|A|^2 = 0$. We set

$$h^1 := \Re h \qquad \text{and} \qquad h^2 := \Im h.$$

Let us prove that h is holomorphic. Using (2.37) and (2.38), we get

$$\begin{cases} h_1^1 = \nu_1 \cdot u_{22} + \nu \cdot u_{122} \\ h_2^2 = \nu_2 \cdot u_{12} + \nu \cdot u_{122} = h_1^1 \\ h_1^2 = \nu_1 \cdot u_{12} + \nu \cdot u_{112} \\ h_2^1 = -\nu_2 \cdot u_{11} - \nu \cdot u_{211} = -h_1^2, \end{cases}$$

which means that h satisfies the Cauchy-Riemann equations. □

2.4 The case $n = 4$ for contact angles close to 0

In this section we exploit the connection between the cones minimizing $\mathcal{A}^\theta$ and the one-homogeneus minimizer of the Alt-Caffarelli functional. In order to understand how this connection arises, consider the case where a cone Ω^θ minimum of $\mathcal{A}^\theta$ is the region below the graph of a one-homogeneous lipschitz function u^θ over the x_{n+1} direction. Namely, calling $M^\theta = \partial\Omega \cap \mathcal{R}_+^{n+1}$, it holds $M^\theta = \{(x', u^\theta(x')) : x' \in \mathbb{R}^n, u^\theta(x') > 0\}$.

Setting $v^\theta = \frac{u^\theta}{\tan(\theta)}$, and assuming that the Lipschitz constant of v^θ, and thus $|\nabla v^\theta|$, is uniformly bounded as θ goes to 0, we can write

$$\begin{aligned} \mathcal{A}^\theta(\Omega^\theta) &= \int_{u^\theta>0} \sqrt{1 + |\nabla u^\theta|^2} - \cos(\theta)\mathbb{1}_{u^\theta>0} \\ &= \int_{v^\theta>0} \Big((1 - \cos(\theta)) \Big) \mathbb{1}_{v^\theta>0} + \frac{1}{2}\tan^2(\theta)|\nabla v^\theta|^2 + O(\tan^4(\theta)) = \\ &= \frac{1}{2}\tan^2(\theta) \int_{v^\theta>0} \Big(|\nabla v^\theta|^2 + \mathbb{1}_{v^\theta>0} + O(\theta^2) \Big), \end{aligned}$$

where we used that $\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$ as x goes to zero. Thus, we can expect that, as θ goes to zero, $v^\theta = \frac{u^\theta}{\tan(\theta)}$ converges in some sense to an one-homogeneous minimizer of the one-phase Alt-Caffarelli functional

$$J(v) = \int_{\mathbb{R}^n} |\nabla v|^2 + \mathbb{1}_{v>0}. \tag{2.40}$$

The definition of a minimizer is the following.

Definition 2.4.1. We say that $u \in H^1_{loc}(\mathbb{R}^n)$ minimizes J if, for any $U \subset\subset \mathbb{R}^n$, and any $w \in H^1(U)$ such that $u - w \in H^1_0(U)$, we have

$$J(u) \leq J(w)$$

Using these idea, we will make use of the following teo known facts about minimizers of J. (see [10] and [3])

Theorem 2.4.2. *Let $u \in H^1_{loc}(\mathbb{R}^n)$ be a minimizer of J in $\mathbb{R}^n$. Then there exists a constant $c_0(n) > 1$ such that*

$$\frac{1}{c_0} d(x, \partial\{u > 0\}) \leq u(x) \leq d(x, \partial\{u > 0\}), \qquad u(x) > 0$$

Theorem 2.4.3. *If $n \leq 4$ and $u \in H^1_{loc}(\mathbb{R}^n)$ is a minimizer of J, then there exists a unit vector n such that*

$$u = (x \cdot n)_+.$$

Having this in mind, we want to prove that, for θ small, a minimizing cone of $\mathcal{A}^\theta$ with an isolated singularity at 0 is graphical.

We start our treatment by proving that a smooth minimum Ω of $\mathcal{A}^\theta$ has a minimizing boundary away from ∂M.

Definition 2.4.4. Let $A \subset \mathbb{R}^{n+1}$ be an open set. We say that a set M is a mass-minimizing boundary in A if there exists a set E with locally finite perimeter such that $\partial^\star E \cap A = M \cap A$, and E is a perimeter minimizer in A.

Definition 2.4.5. Let $\gamma > 0$, and let $x' \in \mathbb{R}^n$. We call the cylinder of center x' and radius γ the set $U_\gamma(x') := B_n(0, \gamma) \times \mathbb{R}$.

More generally, if Γ is a set in $\mathbb{R}^n$, $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the orthogonal projection, we call the γ-cylinder generated by Γ the set $U_\gamma(\Gamma) := \{x \in \mathbb{R}^{n+1} : d(\pi(x), \Gamma)\} < \gamma$

Lemma 2.4.6. *Let Ω be an open set, minimum of $\mathcal{A}^\theta$ in $B(0;r)$ that is smooth in $B(0,r)$. Let $x \in M \cap B(0,r) \cap \mathbb{R}^{n+1}_+$ such that $d(\pi(x), \partial M) > \gamma$. Let C be $B(x, \gamma)$ or $U_\gamma(\pi(x))$.*

Then M is a mass minimizing boundary in $C \cap B(0,r)$.

Proof. By hypothesis, $\overline{C \cap B(0,r) \cap M} \cap \mathbb{R}^n = \emptyset$, thus we can choose $\epsilon > 0$ such that, for any $x \in \overline{C \cap B(0,r) \cap M}$ holds $x_{n+1} \geq \epsilon$. Notice that either Ω or $\mathbb{R}^{n+1}_+ \setminus \overline{\Omega}$ has empty intersection with $C \cap B(0,r) \cap \{x_{n+1} < \epsilon\}$, and we call this set E. Indeed, if we had $x \in \Omega$ and $y \in \mathbb{R}^{n+1}_+ \setminus \overline{\Omega}$ with $x, y \in C \cap B(0,r)$, and $x_{n+1}, y_{n+1} < \epsilon$, then there should be a point z in the segment between x and y satisfying $z \in \partial\Omega$, but, by convexity of $C \cap B(0,r) \cap \{x' : x'_{n+1} < \epsilon\}$, $z \in C \cap B(0,r) \cap M$ and $z_{n+1} < \epsilon$, that, by how we chose ϵ, gives a contradiction. Therefore, E is a minimizer for $\mathcal{A}^\delta$ in $B(0,r)$, where δ is either θ or $\pi - \theta$, and, for any F with locally finite perimeter such that $E \Delta F \subset U \subset\subset C \cap B(0,r)$, we can choose $\epsilon' < \epsilon$ such that the set $F_{\epsilon'}$, defined by cutting F with the upper half-space $\{x_{n+1} > \epsilon'\}$ inside $C \cap B(0,r)$, is a set of locally finite perimeter satisfying $P(F_{\epsilon'};U) \leq P(F;U)$. Moreover, by our choise of ϵ, we still have $F_{\epsilon'}\Delta E \subset U$, and thus,

$$P(E;U) = \mathcal{H}^n(M \cap U) = \mathcal{A}^\delta_U(E) \leq \mathcal{A}^\delta_U(F_{\epsilon'}) = P(F_{\epsilon'};U) \leq P(F;U).$$

□

Remark 2.4.7. Notice that the previous Lemma is true also for Ω smooth cone with an isolated singularity at zero. And notice that the Lemma is true for any ball $B(x,r)$ such that

$$x \in M \qquad B(x,r) \subset\subset \mathbb{R}^{n+1}_+,$$

without any constraint on $d(\pi(x), \partial M)$.

We state now a consequence of the Allard regularity Theorem (see [2]).

Proposition 2.4.8. *There is an $\epsilon_0(n)$ such that the following holds. Let M be a mass minimizing boundary in $B(0,r)$, $r > 0$, such that*

$$0 \in M, \qquad \sup_{M \cap B(0,r)} r^{-1}|x_{n+1}| < \epsilon \leq \epsilon_0.$$

Then there esxists $c(n) > 0$ such that $M \cap B(0,r/2)$ is the graph over the x_{n+1}-direction of a function u, with the estimate

$$r|\nabla^2 u| + |\nabla u| + r^{-1}|u| \leq c(n)\epsilon, \qquad \text{on}\, B(0,r/2). \tag{2.41}$$

We need also an Harnack inequality for harmonic functions on mass-minimizing boundary, due to Bombieri and Giusti [11].

Theorem 2.4.9. *Let M be a mass-minimizing boundary in $B(0,r) \subset \mathbb{R}^n$. Then, there are constants $\sigma(n) \in (0,1)$, $c(n)$ such that if $u \in C^1(B(0,r))$ satisfies*

$$\int_M \nabla u \cdot \nabla\varphi \, d\mathcal{H}^n = 0, \qquad \forall \varphi \in C^1_c(B(0,r)),\, u > 0 \,\text{on}\, M,$$

then

$$\sup_{M \cap B(0,\sigma r)} u \leq c \inf_{M \cap B(0,\sigma r)} u. \tag{2.42}$$

Remark 2.4.10. Notice that in our context there is a natural choice of an harmonic function on a mass minimizing boundaty: indeed, by Lemma 2.4.6, if Ω is a minimizer of $\mathcal{A}^\theta$, then M is a mass minimizing boundary away from ∂M. Moreover, the height x_{n+1} is a strongly harmonic function in M. Indeed,

$$\begin{aligned}&\nabla_M x_{n+1} = e_{n+1} - (e_{n+1} \cdot \nu)\nu,\\ &\Delta_M x_{n+1} = \operatorname{div}(\nabla_m x_{n+1}) = -\operatorname{div}(\nu)(e_{n+1} \cdot \nu) - e_{n+1} \cdot A\nu = 0,\end{aligned}$$

where we used that $A\nu = 0$, and that divν is the mean curvature of M, that is zero by minimality.

The next Lemma rules out a graphical minimizer of $\mathcal{A}^\theta$ having large pieces that stay too close to the boundary, and it is similar to [10][Lemma 3.4].

Lemma 2.4.11. *Let Ω be a minimizer for $\mathcal{A}^\theta$ in $B(0,1)$, and assume that there exists $u : B^n(0,1) \to [0,\infty)$ Lipschitz such that $M \cap B(0,1) = \partial\Omega \cap \mathbb{R}^{n+1}_+ \cap B(0,1) = \operatorname{graph}(u) \llcorner \{u > 0\}$, and such that $\Omega \cap B(0,1)$ is the region below M.*

Then, there exists a constant $\epsilon(n) > 0$, such that if $\sup_{B(0,1/2)} u \le \epsilon\theta$, we must have $u \equiv 0$ on $B(0,1/4)$.

Proof. We want to use a positive Lipschitz function that is zero on $B(0,1/4)$.

Let us define the harmonic radial function

$$\phi(|x|) := |x|^{2-n} - (1/4)^{2-n},$$

and notice that $\phi(1/4) = 0$, and $\phi(r) < 0$ for $r > 1/4$. Define then

$$v(x) = \bar{c}\epsilon\theta \max\{-\phi(|x|), 0\}.$$

where $\bar{c}(n)$ is chosen so that $v > u$ on $\partial B(0,1/2)$. Thus, $v \equiv 0$ on $B(0,1/4)$ and $u > 0$ on $B(0,1) \setminus B(0,1/4)$. Since $\min(u,v) = u$ on $\partial B(0,1/2)$, and therefore, we can define the function

$$\tilde{v}(x) = \begin{cases} \min(u,v)(x), & x \in B(0,1/2),\\ u(x), & x \in B(0,1) \setminus B(0,1/2).\end{cases}$$

Hence, if we take $\tilde{\Omega}$ to be the region below the graph of $\tilde{v}$ in $\mathbb{R}^{n+1}_+$, it holds $\Omega\Delta\tilde{\Omega} \subset\subset B(0,1)$, and therefore $\mathcal{A}^\theta_{B(0,1)}(\Omega) \le \mathcal{A}^\theta_{B(0,1)}(\tilde{\Omega})$, that is, since $v \equiv 0$ on $B(0,1/4)$

$$\int_{B(0,1/2)\cap\{u>0\}} \sqrt{1+|\nabla u|^2} - \cos(\theta)\,dx \le \int_{\{u>0\}\cap B(0,1/2)\setminus B(0,1/4)} \sqrt{1+|\nabla \min(u,v)|^2} - \cos(\theta).$$

Using the inequality $\sqrt{1+a^2}-\sqrt{1+b^2} \le a^2-b^2$ we can write, calling $B(0,r) = B_r$

$$\begin{aligned}
\int_{B_{1/4}} \sqrt{1+|\nabla u|^2} - \cos(\theta)\mathbb{1}_{\{u>0\}} &\le \int_{B_{1/2}\setminus B_{1/4}} \sqrt{1+|\nabla \min(u,v)|^2} - \sqrt{1+|\nabla u|^2} \\
&= \int_{\{u>v\}\cap B_{1/2}\setminus B_{1/4}} \sqrt{1+|\nabla v|^2} - \sqrt{1+|\nabla u|^2} \\
&\le \int_{\{u>v\}\cap B_{1/2}\setminus B_{1/4}} |\nabla v|^2 - |\nabla u|^2 \\
&\le -2 \int_{\{u>v\}\cap B_{1/2}\setminus B_{1/4}} \nabla(u-v)\cdot\nabla(v)
\end{aligned}$$

where we used that $|\nabla v|^2 - |\nabla u|^2 + 2\nabla(u-v)\cdot\nabla v = -|\nabla(u-v)|^2 \le 0$. Now, since v is harmonic in $B_{1/2} \setminus B_{1/4}$, and $u - v \equiv 0$ on $\partial B1/2$, and $v \equiv 0$ on $\partial B_{1/4}$,

$$\begin{aligned}
-2 \int_{\{u>v\}\cap B_{1/2}\setminus B_{1/4}} \nabla(u-v)\cdot\nabla(v) &= 2\int_{B_{1/2}\setminus B_{1/4}} \nabla \max(u-v,0)\cdot\nabla v \\
&2\int_{\partial B_{1/4}} u\partial_r v \\
&= 2c(n)\epsilon\theta \int_{\partial B_{1/4}} u,
\end{aligned}$$

and thus

$$\int_{B_{1/4}} \sqrt{1+|\nabla u|^2} - \cos(\theta)\mathbb{1}_{\{u>0\}} \le 2c(n)\epsilon\theta \int_{\partial B_{1/4}} u. \tag{2.43}$$

The aim is to prove that $\int_{\partial B_{1/4}} u = 0$. This would be enough, indeed, we can apply the result to B_r with $r < 1$, gaining that also $\int_{\partial B_{r/4}} u = 0$, and thus $u \equiv 0$ on $B_{1/4}$.

A simple computation shows that, for $t \geq 0$ and $\theta \in [0, \pi/2]$,
$2\sqrt{1+t^2} - t\theta - 1 \geq \sqrt{4-\theta^2} - 1 \geq 1 - \theta^2/3 \geq 1 - \theta^2/2 + \theta^4/4 \geq \cos(\theta)$, then, using the trace inequality, the bound $|u| \leq \epsilon\theta$ and the above inequality with $t = |\nabla u|$, we can write

$$\begin{aligned}\int_{\partial B_{1/4}} u &\leq c_0(n) \int_{B_{1/4}} |\nabla u| + |u| \\ &\leq c_1(n)\theta^{-1} \int_{B_{1/4}} |\nabla u|\theta + \theta^2 \mathbb{1}_{\{u>0\}} \\ &\leq c_2(n)\theta^{-1} \int_{B_{1/4}} |\nabla u|\theta + (1-\cos(\theta))\mathbb{1}_{\{u>0\}} \\ &\leq 2c_2(n)\theta^{-1} \int_{B_{1/4}} \sqrt{1+|\nabla u|^2} - \cos(\theta)\mathbb{1}_{\{u>0\}},\end{aligned}$$

that, together with the (2.43), implies $\int_{\partial B_{1/4}} u = 0$ for ϵ small enough depending only on the dimension n. □

With the next Lemma we show how an appropriate height bound close to the boundary of M gives a criterion to write M as a Lipschitz graph even far away from the boundary, extending the height bound with different constants.

Lemma 2.4.12. *There exists $\theta_0(n, \gamma) > 0$ such that the following holds.*

Let $\theta \leq \theta_0$, and $\Omega \subset \mathbb{R}^{n+1}_+$ be an open set with smooth boundary in $B(0,1) \cap \overline{\mathbb{R}^{n+1}_+}$, that is a minimizer of $\mathcal{A}^\theta$ in B_1. Let $M := \partial\Omega \cap \mathbb{R}^{n+1}_+$. Assume that there exist $0 < \gamma \leq 1$ and $c_0 > 0$ such that

$$d(0, \partial M) \leq \frac{\gamma}{2}; \tag{2.44}$$

if $x \in M \cap B_1$ with $d(\pi(x), \partial M \cap B_1) < \gamma$ we have

$$\frac{1}{2c_0} \tan(\theta) d(\pi(x), \partial M) \leq x_{n+1} \leq 2\tan(\theta) d(\pi(x), \partial M). \tag{2.45}$$

Then there exists a Lipschitz function $u : B^n_{1/2} \to \mathbb{R}$ and a constant $c(n, \gamma)$ such that

1. *$\tilde{M} := \{(z, u(z)) : z \in B^n_{1/2},\ u(z) > 0\} \subset M$.*

2. *$M \cap B_{3/4} \cap B^n_{1/2} \cap U_{\frac{\gamma}{2}}(\partial M \cap B_1) \subset \tilde{M}$.*

3. *If C is a connected component of $(M \cup \partial M) \cap B_{3/4} \cap \pi^{-1}(B^n_{1/2})$, then $C \subset \tilde{M}$.*

4. *Let $\tilde{\Omega} \subset \mathbb{R}^{n+1}_+$ be the region below the graph of u. Then $\tilde{\Omega} \subset \Omega$. Moreover, $\tilde{M}$ has positive distance, depending only on γ and n, from any other piece of M in $B_{3/4} \cap \pi^{-1}(B^n_{1/2})$*

5. *$d(0, \partial\{u > 0\}) \leq \frac{\gamma}{2}$ and $|u| + \mathrm{Lip}(u) \leq c(n, \gamma)\tan(\theta)$,*

6. *For any $z \in B^n_{1/2}$ with $u(z) > 0$, there exists a constant $c(n, \gamma)$ such that*

$$\frac{1}{c(n,\gamma)}\tan(\theta) d(z, \partial\{u > 0\}) \leq u(z) \leq c(n, \gamma)\tan(\theta) d(z, \partial\{u > 0\}), \tag{2.46}$$

where $\partial\{u > 0\}$ is the boundary of the set $\{u > 0\}$ in $B^n_{1/2}$.

Proof. Notice that, since M has zero mean curvature, and it is smoothly embedded at the boundary of $\mathbb{R}^{n+1}_+$ with fixed contact angle $\theta \in (0, \frac{\pi}{2})$, there exists $\delta << 1$ depending on M, such that $U_\delta(\partial M) \cap M$ is graphical over $\mathbb{R}^n$, and Ω is the region below its boundary. Here we extend this property.

Take any $x \in M \cap B_{3/4} \cap \pi^{-1}(B^n_{1/2}) \cap U_{\frac{\gamma}{2}}(\partial M \cap B_1)$, take $r := \frac{d(\pi(x), \partial M \cap B_1)}{4}$, and notice that, since $d(\pi(x), \partial M \cap B_1) \leq |\pi(x)| + d(0, \partial M \cap B_1) \leq \frac{1}{2} + \frac{\gamma}{2} < 1$, $B(x, r) \subset B(0, 1) \cap U_\gamma(\partial M \cap B_1)$. Then, thanks to Lemma 2.4.6, M is a mass minimizing boundary in $B(x, r)$, and (2.45) gives

$$y_{n+1} \leq 8\tan(\theta) r, \qquad y \in B(x, r).$$

Thus, thanks to Proposition 2.4.8, provided that θ is small enough, depending on n and γ, there exists a C^1 function $u_x : B^n(\pi(x), \frac{r}{2})$, such that

$$M \cap B\left(x, \frac{r}{2}\right) = \mathrm{graph}(u_x), \qquad |\nabla u_x| \leq 8\tan(\theta).$$

Notice that, any other possible $x' \in B_{3/4} \cap M \cap \pi^{-1}(B^n_{1/2})$ such that $\pi(x') = \pi(x)$ must satisfy $x'_{n+1} \leq 8\tan(\theta) r$, thus $|x' - x| \leq 8\tan(\theta) r < \frac{r}{2}$ for θ small, and $B(x, r/2) \cap M$ is a graph, hence $x' = x$. This proves that the functions u_x paste well together, and we can define a Lipschitz function u such that

$$M \cap B_{3/4} \cap \pi^{-1}(B^n_{1/2}) \cap U_{\frac{\gamma}{2}}(\partial M \cap B_1) = \mathrm{graph}(u), \qquad |\nabla u| \leq 8\tan(\theta).$$

Let now $\sigma < \frac{1}{4}$ and C like in Theorem 2.4.9, and take any

$$x \in \operatorname{graph}(u) \cap B_{3/4} \cap \pi^{-1}(B^n_{1/2}) \cap \partial U_{\frac{\gamma}{4}}(\partial M \cap B_1).$$

M is a mass minimizing boundary in $B(x, \sigma\gamma)$, and we can thus apply the Harnack inequality, obtaining, thanks to (2.45) considered at $d(\pi(\cdot), \partial M \cap B_1) = \frac{\gamma}{4}$,

$$\frac{1}{2C_0 C}\tan(\theta)\frac{\gamma}{4} \leq y_{n+1} \leq 2C\tan(\theta)\frac{\gamma}{4}, \qquad y \in B(x, \sigma^2\gamma) \cap M.$$

Applying again Proposition 2.4.8, provided that θ is small enough, we can extend graph(u) in M until $B_{3/4} \cap \pi^{-1}(B^n_{1/2}) \cap \partial U_{\frac{\gamma}{4}+\sigma^2\gamma}(\partial M \cap B_1)$, with u satisfying

$$|\nabla u| \leq \tan(\theta)\frac{C}{2\sigma^2}.$$

Moreover, by the same argument as above, there are no points of M below the graph of u. We continue this process other N times, where N is the first integer such that $\frac{\gamma}{4} + \sigma^2\gamma + N\sigma^2\frac{\gamma}{2} > \frac{3}{4}$, extending u to a Lipschitz function satisfying

$$|\nabla u| \leq c(n,\gamma)\tan(\theta), \qquad c(n,\gamma) = \frac{C^{N+1}}{2\sigma^2},$$

$$\frac{1}{2c_0 c(n,\gamma)}\tan(\theta) d(z, \partial M) \leq d(z, \partial M) \leq c(n,\gamma)\tan(\theta) d(z, \partial M).$$

Notice that, since the number of steps N is depending on γ, n only, and at each step we chose θ small enough depending on $\frac{C^i}{2\sigma^2}$, where $i \leq N+1$, we can choose a $\theta_0(n,\gamma)$ such that for any $\theta < \theta_0$ the argument above works.

Note that, by construction, there are no points of M below the graph of u, and, since close to the boundary ∂M Ω must be the region below the graph of u, if $\tilde{\Omega}$ is the open set in the statement, then $\tilde{\Omega} \subset \Omega$.

Moreover, by construction, any other point of $M \cap B_{3/4} \cap \pi^{-1}(B^n_{1/2})$ that doesn't lay on the graph of u, that we call $\tilde{M}$, must satisfy

$$d(x, \tilde{M}) > \frac{\sigma^2\gamma}{2}.$$

All of this imply that any connected component of $M \cap B_{3/4} \cap \pi^{-1}(B^n_{1/2})$ that touches $\partial M \cap B_{3/4} \cap \pi^{-1}(B^n_{1/2})$ must be contained in $\tilde{M}$. Notice that u is not defined in all points of $B^n_{1/2}$. The points z in which u is not defined are points z such that, if $x \in M \cap B_{3/4} \cap \pi^{-1}(z)$, then the connected component of $M \cap B_{3/4} \cap \pi^{-1}(z)$ doesn't touch $\partial M \cap B_{1/2}$, so we can set $u(z) = 0$. Notice also that $\partial\{u > 0\} = \partial M \cap B^n_{1/2}$, thus (2.46) follows by construction.

□

Remark 2.4.13. If Ω is a smooth minimizer for $\mathcal{A}^\theta$, M has zero mean curvature, and meets $\mathbb{R}^n$ with a fixed contact angle θ. It can be proved that this conditions allow to write, in small balls centered ∂M, Ω as the region below the graph of a Lipschitz function over $\mathbb{R}^n$, and that, for some small γ, the hypothesis of the previous Lemma are always satisfied.

We show now what happens when we apply the previous theorem to a sequence of minimizers satisfying the assumptions of the previous Lemma with the same γ and c_0, and such that $\theta \to 0$. We need first the definition of local Hausdorff convergence.

Definition 2.4.14. Suppose that X_i is a sequence of closed sets in $\mathbb{R}^n$, and Ω is an open set in $\mathbb{R}^n$. We say that X_i converges in the local Hausdorff distance in Ω to the closed set X, if for every compact set $K \subset \Omega$, and every open set U such that $K \subset U \subset \Omega$, we have

$$\lim_{i\to\infty} \operatorname{dist}_{K,U}(X_i, X) = 0,$$

where, for any pair of closed subsets X, Y of Ω, we define

$$\operatorname{dist}_{K,U}(X,Y) := \max\{\max_{x\in X\cap K} \operatorname{dist}(x, Y\cap U), \max_{y\in Y\cap K} \operatorname{dist}(y, X\cap U)\}$$

Remark 2.4.15. It can be proven that, if $X_i \to X$ in the local Hausdorff distance in Ω, then

$$d(\cdot, X_i) \to d(\cdot, X), \qquad \text{in } L^\infty_{loc}(\Omega).$$

Proposition 2.4.16. *Let $\theta_i > 0$ be a sequence with $\lim_{i\to\infty} \theta_i = 0$, and let $\Omega_i \subset \mathbb{R}^{n+1}_+$ be a sequence of open set with smooth boundary in B_1, such that Ω_i is a minimizer for $\mathcal{A}^{\theta_i}$ in B_1, and let, as usual, $M_i := \partial\Omega \cap \mathbb{R}^{n+1}_+$.*

Assume that, for some fixed $\gamma, c_0 > 0$, (2.44) and (2.45) are satisfied by each M_i.

Let $u_i : B^n_{1/2} \to [0,\infty)$ be the function obtained in Lemma 2.4.12. Then, up to a subsequence,

$$v_i = \frac{u_i}{\tan(\theta_i)}$$

converges in $(W^{1,2}_{loc} \cap C^\alpha_{loc})(B^n_{1/2})$, for any $\alpha < 1$, to a Lipschitz function v that minimizes the Alt-Caffarelli functional J (2.40) in $B^n_{1/2}$.

Moreover, $\partial\{v_i > 0\} \to \partial\{v > 0\}$ in the local Hausdorff distance in $B^n_{1/2}$, and $\mathbb{1}_{\{v_i>0\}} \to \mathbb{1}_{\{v>0\}}$ in $L^1_{loc}(B^n_{1/2})$.

Proof. Since, by Lemma 2.4.12, $|v_i| + Lip(v_i) \leq c(n,\gamma)$, by Ascoli-Arzelà Theorem, up to a subsequence, there exists a function $v : B^n_{1/2} \to [0,\infty)$, such that $v_i \to v$ in $L^\infty_{loc}(B^n_{1/2})$ and in $C^\alpha_{loc}(B^n_{1/2})$, and v satisfies $|v| + Lip(v) \leq c(n,\gamma)$. In particular, the convergence is strong in L^2_{loc}, and, up to take another subsequence, $\nabla v_i \rightharpoonup \nabla v$ weakly in L^2, and, by lower semicontinuity of the L^2-norm of the gradient,

$$\|\nabla v\|_2 \leq \liminf_{i\to\infty} \|v_i\|_2.$$

Being $W^{1,2}$ an Hilbert space, the strong convergence in $W^{1,2}$ will be proved once we know that $\limsup_{i\to\infty} \|\nabla v_i\|_2 \leq \|\nabla v\|_2$.

Before proving this, we prove the local Hausdorff convergence of the free boundaries, that will turn out to be useful to prov the strong convergence of the functions.

Let us fix a compact set $K \subset B^n_{1/2}$, and an open set $U \subset B^n_{1/2}$, with $K \subset U$. Fix $\epsilon > 0$ small enough, and cover $K \cap \partial\{v > 0\}$ with a finite number of balls $(B(x_j,\epsilon))_{j=1}^k$, such that $x_j \in K \cap \partial\{v > 0\}$, $B(x_j,\epsilon) \subset U$, such that there exist $y_j \in B(x_j,\epsilon)$, with $0 < v(y_j) < \epsilon$ and, provided that ϵ is small enough, such that $B(y_j, 2c(n,\gamma)\epsilon) \subset U$. Choose $i_0 \geq 0$ such that, for $i \geq i_0$, $0 < v_i(y_j) < \epsilon$, for any $j = 1, \cdot k$. Taking now $x \in K \cap \partial\{v > 0\}$, there exists j such that $x \in B(x_j,\epsilon)$, thus $d(x, \partial\{v_i > 0\} \cap U) \leq \epsilon + d(y_j, \partial\{v_i > 0\} \cap U)$. By Lemma 2.4.12, $d(y_j, \partial\{v_i > 0\}) \leq c(n,\gamma)v_i(y_j) < c(n,\gamma)\epsilon$, thus, since $B(y_j, 2c(n,\gamma)\epsilon) \subset U$, $d(y_j, \partial\{v_i > 0\} \cap U) = d(y_j, \partial\{v_i > 0\}) \leq c(n,\gamma)\epsilon$, and we proved that

$$\max_{x\in K\cap\partial\{v>0\}} d(x, \partial\{v_i > 0\} \cap U) \to 0, \qquad \text{as}\, i \to \infty$$

Conversely, take $x_i \in \partial\{v_i > 0\} \cap K$ such that $d(x_i, \partial\{v > 0\} \cap U) = \max_{x\in\partial\{v_i>0\}\cap K} d(x, \partial\{v > 0\} \cap U)$. Using the Uhryson Lemma, is sufficient to take a subsequence and assume that $x_i \to \bar{x} \in K$. Since $\bar{x} \in K \subset U$, it is sufficient to prove that $\bar{x} \in \partial\{v > 0\}$, and we will have

$$\max_{x\in\partial\{v_i>0\}\cap K} d(x, \partial\{v > 0\} \cap U) \leq d(x_i, \bar{x}) \to 0, \qquad \text{as}\, i \to \infty.$$

If $\bar{x} \notin \partial\{v > 0\}$, then there exists $r > 0$ such that $v \equiv 0$ on $B(\bar{x}, 2r) \subset\subset B^n_{1/2}$. Then $v_i = \frac{u_i}{\tan(\theta)} \to 0$ uniformly on $B(\bar{x}, 2r)$, and, by Lemma 2.4.11, for i big enough $v_i \equiv 0$ on $B(\bar{x}, r)$. Thus, $x_i \notin \partial\{v_i > 0\}$, that is a contradiction.

We claim now that

$$\lim_{i\to\infty} \mathbb{1}_{\{v_i>0\}} = \mathbb{1}_{\{v>0\}}, \qquad \text{in}\, L^1_{loc}(B^n_{1/2}).$$

Indeed, let K be a compact subset of $B^n_{1/2}$, then

$$\begin{aligned}\|\mathbb{1}_{\{v_i>0\}} - \mathbb{1}_{\{v>0\}}\|_{L^1(K)} = \|\mathbb{1}_{\{v_i>0\}} - 1\|_{L^1(K\cap\{u>0\})} \\ + \|\mathbb{1}_{\{v_i>0\}}\|_{L^1(K\cap\{u=0\}^o)} \\ + \|\mathbb{1}_{\{v_i>0\}}\|_{L^1(K\cap\partial\{v>0\})}.\end{aligned}$$

The first and the second term converge to zero by dominated convergence, thanks to the uniform convergence of v_i to v on K. It is then sufficient to prove that $|\partial\{v > 0\}| = 0$. By Lebesgue's differentiation theorem, almost every $x \in \partial\{v > 0\}$ has Lebesgue density equal to 1, i.e.

$$\lim_{r\to 0} \frac{|\partial\{v > 0\} \cap B(x, r)|}{\omega_n r^n} = 1,$$

where $\omega_n := |B^n_1|$. Take such an x, and take r such that $(1 - 2^{-k})\omega_n r^n < |B(x, r)\cap\partial\{v > 0\}|$, with $4 < k \in \mathbb{N}$ to be chosen properly. Take $y \in B(x, r/8)$ with $v(y) > 0$, and notice that, for any $z \in B(y, r/4)$, $B(z, r2^{-k}) \subset B(x, r)$. Then, we can find some $z' \in B(z, r2^{-k}) \cap B(x, r) \cap \partial\{v > 0\}$, and we have

$$v(z) = v(z) - v(z') \leq c(n, \gamma)|z - z'| < c(n, \gamma)r2^{-k}, \qquad z \in B(y, r/4).$$

Thus, if k is big enough, Lemma 2.4.11 and the uniform convergence of v_i to v imply that $v(y) = 0$. This means that there isn't an $x \in \partial\{v > 0\}$ with Lebesgue density 1, and therefore $|\partial\{v > 0\}| = 0$.

We prove now together that v is a minimizer and that $\limsup_{i\to\infty} \|\nabla v_i\|_{L^2(K)} \leq \|\nabla v\|_{L^2(K)}$, for any compact set $K \subset B^n_{1/2}$.

Take $0 < r < \frac{1}{2}$, and take $\eta \in C^\infty_c(B^n_r)$, $0 \leq \eta \leq 1$. Let us define

$$\alpha_i := v\eta + (1 - \eta)v_i,$$

and notice that α_i and v_i have the same trace on $\partial B^n_{1/2}$. Thus, as we did in previous proofs, we can make a comparison between Ω_i and the subgraph of $\theta_i\alpha_i$, and thanks to the minimality of Ω_i with respect to $\mathcal{A}^{\theta_i}$, we get

$$\int_{B^n_r} \left(\sqrt{1 + \tan^2(\theta_i)|\nabla v_i|^2} - \cos(\theta_i)\right)\mathbb{1}_{\{v_i>0\}} \leq \int_{B^n_r} \left(\sqrt{1 + \tan^2(\theta_i)|\nabla \alpha_i|^2} - \cos(\theta_i)\right)\mathbb{1}_{\{\alpha_i>0\}}. \tag{2.47}$$

Notice that, thanks to Lemma 2.4.12, $\tan(\theta_i)\alpha_i$ gives a small perturbation of $\tilde{M}_i$, in the sense that, for i big enough, the pieces of $\Omega_i \cap B_{3/4} \cap B^n_{1/2} \setminus \Omega_i$ have positive distance, depending only on γ, from the region below graph$(\tan(\theta_i)\alpha_i)$, and (2.47) is then a rigorous application of minimality of Ω_i, besides the region below the graph of u_i doesn't cover the whole part of Ω_i that lays in $B_{3/4} \cap \pi^{-1}(B^n_{1/2})$.

Going back to the proof, if we use that $\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2)$, and that $|\nabla v_i| \leq c(n,\gamma)$, we get

$$\int_{B^n_r} \tan^2(\theta)\frac{|\nabla v_i|^2}{2} + (1-\cos(\theta))\mathbb{1}_{\{v_i>0\}}\,dx$$
$$\leq O(\theta_i^4) + \int_{B^n_r} \tan^2(\theta_i)\frac{|\nabla \alpha_i|^2}{2} + (1-\cos(\theta))\mathbb{1}_{\{\alpha_i>0\}} + O_\eta(\theta_i^4)\,dx,$$

where, with the notation O_η we are underlyining the dependence of the O on η, that is not a big deal, since we are now taking η fixed. We can divide the inequality by $\tan^2(\theta)$, getting

$$\int_{B^n_r} |\nabla v_i|^2 + \mathbb{1}_{\{v_i>0\}}\,dx \leq o(1) + \int_{B^n_r} |\nabla \alpha_i|^2 + \mathbb{1}_{\{\alpha_i>0\}}\,dx,$$

where $o(1)$ is considered as i goes to ∞. Now, choose $\eta = \eta_\delta$ such that $\eta_\delta = 1$ on $B^n_{r(1-\delta)}$, and $\eta \equiv 0$ on $B^n_r \setminus B^n_{r(1-\delta/2)}$. With this choice, $\alpha_i = v$ on $B^n_{r(1-\delta)}$, thus, estimating with 1 the indicator function of $\{\alpha_i > 0\}$ on $B^n_r \setminus B^n_{r(1-\delta)}$, we get

$$\int_{B^n_r} |\nabla v_i|^2 + \mathbb{1}_{\{v_i>0\}}\,dx \leq o(1) + \int_{B^n_r} |\nabla \alpha_i|^2\,dx$$
$$+ \int_{B^n_{r(1-\delta)}} \mathbb{1}_{\{v>0\}}\,dx + \omega_n r^n(1-(1-\delta)^n)$$

Since $|\nabla\alpha_i|^2 = |\nabla\eta(v-v_i)+\eta\nabla v+(1-\eta)\nabla v_i|^2$, and since $v_i \to v$ uniformly on B^n_r, and $|\nabla v_i|, |\nabla v| \leq c(n,\gamma)$, we have

$$\int_{B^n_r} |\nabla v_i|^2 + \mathbb{1}_{\{v_i>0\}}\,dx \leq o(1)+$$

$$\int_{B_r^n} \eta^2 |\nabla v|^2 + 2c(n,\gamma)(1-\eta)\, dx.$$

$$+ \int_{B^n_{r(1-\delta)}} \mathbb{1}_{\{v>0\}}\, dx + \omega_n r^n (1-(1-\delta)^n).$$

Taking the lim sup as $i \to \infty$ before, and then the limit as $\delta \to 0$, we finally get

$$\limsup_{i\to\infty} \int_{B_r^n} |\nabla v_i|^2 \le \int_{B_r^n} |\nabla v|^2.$$

Take now $w \in H^1(B_r^n)$, such that $v - w \in H_0^1(B_r^n)$. Since w has positive trace on ∂B_r^n, it holds $J(w_+) \le J(w)$, hence we can assume that $w \ge 0$. Then, if we repeat the same proof as above, with $w_i := \eta_\delta w + (1-\eta_\delta) v_i$ in place of α_i, getting

$$\int_{B_r^n} |\nabla v_i|^2 + \mathbb{1}_{\{v_i>0\}}\, dx \le o(1) + \int_{B_r^n} |\nabla w_i|^2\, dx$$

$$+ \int_{B^n_{r(1-\delta)}} \mathbb{1}_{\{w>0\}}\, dx + \omega_n r^n (1-(1-\delta)^n).$$

The difference now is that we now $v_i \to v$ in $H^1(B_r^n)$, and thus $\nabla w_i \to \nabla(\eta_\delta(w-v)+v)$ in $L^2(B_r^n)$, that yields to

$$\int_{B_r^n} |\nabla v|^2 + \mathbb{1}_{\{v>0\}}\, dx \le$$

$$\int_{B_r^n} |\nabla(\eta_\delta(w-v)+v)|^2\, dx$$

$$+ \int_{B^n_{r(1-\delta)}} \mathbb{1}_{\{w>0\}}\, dx + \omega_n r^n (1-(1-\delta)^n).$$

Using finally that we can choose η_δ such that $|\nabla \eta_\delta| \lesssim \frac{1}{r\delta}$, and that $w - v \in H_0^1(B_r^n)$, we can take the limit as $\delta \to 0$, obtaining

$$J(v) \le J(w).$$

□

Remark 2.4.17. We remark that, in order to be really precise, in order to make rigorous the comparison between w_i and v_i in the previous proof, we should have taken first $\varphi \in C_c^\infty(B_r^n)$ and $v + \varphi$ in place of w, then use that $w - v \in H_0^1(B_r^n)$ and take the limits $\varphi \to w - v$. This is because we want that the subgraph of $\tan(\theta_i)w_i$ coincide with Ω_i outside $B_{1/2}$ for θ_i small.

Now we turn our attention into smooth cones with an isolated singularity.

Proposition 2.4.18. *Let $2 \leq n \leq 6$, and let $c_0(n)$ be the constant in Theorem 2.4.2. There exist constants $\theta_0(n), d_0(n) > 0$ such that, if $\theta \in (0, \theta_0)$, and $\Omega \subset \mathbb{R}_+^{n+1}$ is a smooth cone with an isolated singularity at zero, that is a minimizer for $\mathcal{A}^\theta$, then*

$$\frac{1}{2c_0}\tan(\theta)d(\pi(x), \partial M) < x_{n+1} < 2c_0\tan(\theta)d(\pi(x), \partial M), \tag{2.48}$$

for all $x \in \tilde{M} \cap V$ with $d(\pi(x), \partial M) < d_0$, where

$$\tilde{M} := \{x \in M : x_{n+1} = \min_{y \in M \cup \partial M : \pi(y) = \pi(x)} \{y_{n+1}\}\},$$

and $V := B_2^n \setminus \overline{B_{1/2}^n} \times (0, \infty)$.

Proof. Suppose otherwise. Then there are $\theta_i \to 0$, Ω_i cones minimizing $\mathcal{A}^{\theta_i}$, so that one of the inequalities in (2.48) fails for some $x_i \in \tilde{M} \cap \mathcal{S}^n$ with $d(\pi(x_i), \partial M \cap V) =: d_i \to 0$. Since $2 \leq n \leq 6$, thanks to the Simons' Theorem, M doesn't have connected components in $\mathbb{R}^{n+1} \setminus \{0\} \cap \mathcal{S}^n$ with positive distance from $\mathbb{R}^n$. Thus, since M is a minimal hypersurface in $2V$ meeting $\mathbb{R}^n$ at a constant angle θ_i, (2.48) must hold in some neighbourhood of $\partial M \cap V$. Thus, we may choose d_i as above such that (2.48) holds for any $x \in \tilde{M} \cap V$ with $d(\pi(x), \partial M) < d_i$, and such that $\tilde{M} \cap U_{2d_i}(\partial M) \cap V$ is contained in the graphic over $\mathbb{R}^n$ of a Lipschitz function defined on $\pi(V) \cap U_{2d_i}(\partial M)$. Let us define $\Omega_i' := \frac{(\Omega_i - \pi(x_i))}{d_i}$, $M_i' := \partial\Omega_i' \cap \mathbb{R}_+^{n+1} = \frac{(M_i - \pi(x_i))}{d_i}$, and denote $x_i' := \frac{x_i - \pi(x_i)}{d_i}$. Notice that, since x_i lays on the graph of a continuous function u_i such that (2.48) holds for any point y in its graph with $d(\pi(y), \partial M) < d_i$, $(x_i)_{n+1} \leq 2\tan(\theta_i)d_i$. We can choose also $x_i \in S^n$, because $\tilde{M}_i$ is homogeneus, and the height bound is invariant on the radii. Since $x_i \in S^n$, this implies that $|\pi(x_i)| \geq 1 - 4\tan^2(\theta_i)d_i^2$ is far from zero. Then, the singularity of M_i', that is $-\frac{\pi(x_i)}{d_i}$, converges to infinity as i goes to infinity. Notice that, since we are applying an horizontal translation, we are sending points of minimal height of M_i in points of minimal height of M_i'. Let us call $\tilde{M'}_i$ the set of points at minimal height of $M_i' \cup \partial M_i'$ that lays in M_i'. Moreover, $x \in V$ if and only if $x' = \frac{x-\pi(x)}{d_i}$ is such that $2 > |d_i x' + \pi(x_i)| > \frac{1}{2}$.

Thus, (2.48) holds for any $x \in \tilde{M'}_i$ such that

$$d(\pi(x'), \partial M_i') < 1,$$
$$2 > |d_i x' + \pi(x_i)| > \frac{1}{2},$$

and fails at x_i'. Notice also that $d(0, \partial M_i') = 1$. For any $R > 0$, if i is big enough, We can apply Lemma 2.4.12 and Proposition 2.4.16 in $B(0, R)$. With a dyagonal argument, up to a subsequence, we obtain a sequence of Lipschitz functions $u_i : \mathbb{R}^n \to \mathbb{R}$ such that $\frac{u_i}{\tan(\theta_i)}$ converges to an entire minimizer of the Alt Caffarelli functional v uniformly on compact subsets of $\mathbb{R}^n$, $\partial\{u_i > 0\} \to \partial\{v > 0\}$ in the local Hausdorff distance in $\mathbb{R}^n$. Moreover, for any $R > 0$ and $i > i_0(R)$, $\tilde{M'}_i \cap B(0, R)$ is contained in the graph of u. By Theorem 2.4.2, and by Remark 2.4.15, we deduce that, for $i \geq i_0$ big enough, and for any $z : u_i(z) > 0$,

$$\frac{1}{2c_0} \tan(\theta_i) d(z, \partial\{u_i > 0\}) < u_i(z) < 2\tan(\theta_i) d(z, \partial\{u_i > 0\}), \qquad z \in B_1^n.$$

Notice that $x_i' := \frac{x_i - \pi(x_i)}{d_i}$ satisfies

$$\pi(x_i') = 0, \qquad (x_i')_{n+1} \leq 2\tan(\theta_i).$$

Thus, for i big enough, $x_i' \in B(0, 1)$. This means, scaling back, that (2.48) holds for x_i, that is a contradiction. □

We have now all the ingredients necessary to prove the following Theorem.

Theorem 2.4.19. *Let $2 \leq n \leq 6$, and let $\theta_i \to 0$, Ω_i be a sequence of cones minimizing $\mathcal{A}^{\theta_i}$ with an isolated singularity at 0, and $M_i = \partial\Omega_i \cap \mathbb{R}^{n+1}_+$. Then, for i sufficiently large, M_i is contained in the graph of a Lipschitz function u_i over $\mathbb{R}^n$, and $M_i = \mathrm{graph}(u_{\restriction}\{u > 0\})$.Moreover, up to a subsequence, $\frac{u_i}{\tan(\theta_i)}$ converges in $(W^{1,2}_{loc} \cap C^\alpha)(\mathbb{R}^n)$ to an entire minimizer v to the Alt-Caffarelli functional J for all $\alpha \in (0, 1)$, and the free-boundaries $\partial\{u_i > 0\} \to \partial\{v > 0\}$ in the local Hausdorff distance.*

Proof. Let $\theta_0(n)$, $d_0(n)$ given by 2.4.18, If i is such that $\theta_i < \theta_0$, we know that any $x \in \tilde{M}_i \cap V \cap U_{d_0}(\partial M)$ satisfy (2.48), where $\tilde{M}_i$ and V are the same as in the statement of Proposition 2.4.18. Let V' be $B^n_{3/2} \setminus \overline{B^n_{3/4}}$ Repeating the proof of Lemma 2.4.12, using that d_0 depends only on n, and that any connected component of $M_i \cap V'$ touches $\partial M_i \cap V'$, we can find a Lipschitz function $u_i : V' \cap \mathbb{R}^n \to \mathbb{R}$, such that

$$M_i \cap V' \cap \mathbb{R}^{n+1}_+ = \operatorname{graph}(u) \llcorner \{u_i > 0\},$$
$$\partial M_i \cap V' = \partial\{u_i > 0\},$$
$$\frac{1}{c(n)} \tan(\theta) d(z, \partial\{u_i > 0\}) \le u_i(z) \le c(n) d(z, \partial\{u_i > 0\}), \qquad z \in \{u_i > 0\}.$$

In particular, if θ_0 is small enough, $M_i \cap V'$ lays at height smaller than $\frac{1}{1000}$, so that, being $C := B_{5/4} \setminus \overline{B_{4/5}}$, since any point of $M_i \cap C$ is connected to some point in $M_i \cap C \cap V'$, we must have $M_i \cap C \subset V'$. We proved that $M_i \cap C$ is contained in the graph of u_i, and by homogeneity, u_i can be extended to the whole $\mathbb{R}^n$, so that $M_i = \operatorname{graph}\{u_i\} \llcorner \{u_i > 0\}$. Now, the same proof of Proposition 2.4.16 can be applied, so that we get the thesis. □

At this point, in order to prove Theorem 2.2.3 in the case of θ close to zero, Chodosh, Edelen and Li take any sequence of minimizers Ω_i for $\mathcal{A}^{\theta_i}$. Thanks to Theorem 2.4.19, up to a subsequence, M_i are the graph of Lipschitz functions u_i over $\mathbb{R}^n$, such that $v_i = \frac{u_i}{\tan(\theta_i)}$ coverges to a one-homogeneus minimizer of the Alt-Caffarelli functional v. Since $n = 4$, thanks to Theorem 2.4.3, there exists some unit vector ω such that $v = (x \cdot \omega)_+$. They improve this convergence through elliptic estimates, and, thanks to the connection between $\nabla^2 u_i$ and A_{M_i}, they are able to conclude that

$$\lim_{i \to \infty} \frac{|A_{M_i}|}{\tan(\theta_i)} = 0, \qquad \text{uniformly on any} K \subset \mathbb{R}^{n+1}_+ \text{ compact.}$$

Once they have this uniform convergence, they use the Simons'equation and estimates on $\left| \frac{\partial |A_{M_i}|^2}{\tan(\theta)_i} \right|$, in order to conclude that, for i big enough, $|A_{M_i}| = 0$. We refer to [2] for the details.

We chose a different path, that we present in the next section.

2.5 Alternative proof for $n = 3$, and alternative proof for $n = 4$ with θ close to 0

Here we use ideas similar to the ones of Jerison and Savin in [3]. We were able to deal with the case $n = 3$, but, for a matter of time, we were not able to conclude the case $n = 4$. However, in view of the analogies with [3], we think that we just have to take the analogue of their competitor also in dimension $n = 4$, and to check that it works.

At first, we give an instability critherion.

Proposition 2.5.1. *Let $E \subset \mathbb{R}^{n+1}_+$ be a smooth minimizing cone. Suppose that $k \in \mathbb{R}$ and that $c \in C^\infty_c(\mathbb{R}^{n+1} \setminus \{0\})$ is a k-homogeneous non-negative function satisfying*

$$\Delta_M c + |A|^2 c \geq \frac{\Lambda}{|x|^2} c \quad \text{in } M, \qquad \cot(\theta) c\eta \cdot A\eta - \nabla_M c \cdot \eta \geq 0 \quad \text{on } \partial M,$$

for some $\Lambda \in \mathbb{R}$. If

$$\Lambda > \frac{(2-n-2k)^2}{4},$$

then $c \equiv 0$.

Proof. Take a test function $0 \leq \varphi \in C^\infty_c(\mathbb{R}^{n+1} \setminus \{0\})$. An integration by parts gives

$$\int_M \nabla_M c \cdot \nabla_M \varphi - |A|^2 c\varphi \leq -\int_M \frac{\Lambda}{|x|^2} c\varphi + \int_{\partial M} \cot(\theta)\eta \cdot A\eta c\varphi.$$

Take now $0 \leq f \in C^\infty(\mathcal{S}^n)$, $0 \leq h \in C^\infty_c(0,\infty)$ with $\int_0^\infty h > 0$, and let $\varphi(\omega r) = f(\omega)h(r)$, where $\omega \in \mathcal{S}^n$ and $r > 0$. Using the coarea formula and the homogeneity of the functions involveed, by the previous inequality we get

$$\begin{aligned}
&\int_0^\infty r^{n-1} \int_\Sigma r^{k-2} h(r) \nabla_\Sigma c(\omega) \cdot \nabla_\Sigma f(\omega) + r^{k-1} \nabla_M c(\omega) \cdot \omega h'(r)\, d\mathcal{H}^n(\omega)\, dr \\
&+ \int_0^\infty r^{n-1} \int_\Sigma -r^{k-2} |A|^2(\omega) c(\omega) f(\omega) h(r) + \Lambda r^{k-2} c(\omega) f(\omega) h(r)\, d\mathcal{H}^n(\omega)\, dr \\
&\leq \int_0^\infty r^{n-2} \int_{\partial\Sigma} r^{k-1} \cot(\theta) \eta \cdot A\eta c(\omega) f(\omega) h(r)\, d\mathcal{H}^n(\omega)\, dr.
\end{aligned}$$

Notice now that, being c k-homogeneus, $\nabla_M c(\omega) \cdot \omega = kc(\omega)$. In order to have the same radial in any term of the inequality, we integrate by parts $\int_0^\infty kr^{n+k-2} h'(r)\, dr = -\int_0^\infty k(n+k-2) h(r) r^{n+k-3}\, dr$, and we divide by $\int_0^\infty h(r) r^{n+k-3}\, dr$, thus

$$\int_\Sigma \nabla_\Sigma c \cdot \nabla_\Sigma f - |A|^2 cf - \int_{\partial\Sigma} \cot(\theta)\eta \cdot A\eta cf \leq \Big(k(n+k-2) - \Lambda\Big) \int_\Sigma cf.$$

Taking $f = c$, and noting that $k(n+k-2) - \Lambda < -\frac{(n-2)^2}{4}$, if c were non zero we would have a contradiction by (2.22). □

Remark 2.5.2. Notice that, for $1 \geq \alpha > 0$,

$$\begin{aligned}\Delta_M|A|^\alpha + |A|^\alpha|A|^2 &= \alpha\mathrm{div}\left(|A|^{\alpha-2}\frac{\nabla_M|A|^2}{2}\right) + |A|^{\alpha+2}\\ &= \alpha|A|^{\alpha-2}\frac{\Delta_M|A|^2}{2}\\ &\quad+\alpha\left(\alpha-2\right)|A|^{\alpha-2}|\nabla_M|A||^2 + |A|^{\alpha+2}.\end{aligned}$$

Fix now $\lambda \in (0,1)$. A consequence of the Simons' inequality (see 2.65), together with the last computation, implies that $c = |A|^\alpha$ satisfies

$$\begin{aligned}\Delta_M|A|^\alpha + |A|^2|A|^\alpha \geq\ &|A|^{\alpha+2}(1-\alpha)\\ &+2\lambda\alpha\frac{|A|^\alpha}{|x|^2}\\ &\alpha|A|^{\alpha-2}|\nabla_M|A||^2\left(\alpha-2+\lambda+(1-\lambda)\left(1+\frac{2}{n}\right)\right).\end{aligned}$$

Thus, if $n = 3$, $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{4}$, we get

$$\Delta_M|A|^\alpha + |A|^2|A|^\alpha \geq \frac{1}{4}\frac{|A|^\alpha}{|x|^2}. \tag{2.49}$$

Notice that $|A|^\alpha$ is homogeneus of degree $-\alpha = -\frac{1}{2} =: k$. Whith $n = 3$, and letting $\Lambda = \frac{1}{4}$,

$$\Lambda > \frac{(2-n-2k)^2}{4} = 0.$$

Thus, for $\alpha = \frac{1}{2}$, $|A|^\alpha$ satisfies the interior inequality on M in the hypothesis of Proposition 2.5.1.

Notice that $|A|^\alpha$ is not smooth and compactly supported, but, with an argument similar to the one in proof of Theorem 1.3.10, we can make the proof of Proposition 2.5.1 work also for $c = |A|^\alpha$.

What is missing for $|A|^\alpha$ is the boundary inequality, to which will be devoted the rest of this section.

Lemma 2.5.3 (Sign of curvature at the boundary)**.** *Let Ω be a smooth cone that is a minimizer of (2.1). If $e_{n+1}\cdot\nu \neq 0$ on the whole M, then, at any point $x_0 \in \partial M$, holds*

$$\eta \cdot A\eta \geq 0$$

Proof. Let us consider the function $|\nabla_M x_{n+1}|^2$, that is well defined on the whole M. We have

$$\nabla_M x_{n+1} = e_{n+1} - (e_{n+1} \cdot \nu)\nu,$$

then

$$|\nabla_M x_{n+1}|^2/2 = (1 - (e_{n+1} \cdot \nu)^2)/2,$$

and so

$$\begin{aligned}\nabla_M |\nabla_M x_{n+1}|^2/2 &= -(e_{n+1} \cdot \nu)\nabla_M (e_{n+1} \cdot \nu) \\ &= -(e_{n+1} \cdot \nu) A e_{n+1}.\end{aligned}$$

Notice that, at ∂M, $Ae_{n+1} = A\eta(e_{n+1} \cdot \eta) = -\sin(\theta)A\eta$, since $e_{n+1} \perp T_{\partial M}$, and $A\nu = 0 = A(x)x$. Therefore, at ∂M,

$$\partial_\eta |\nabla_M x_{n+1}|^2/2 = \sin(\theta)\cos(\theta)\eta \cdot A\eta,$$

and we just need to prove that $\partial_\eta |\nabla_M x_{n+1}|^2/2 \geq 0$.

A straightforward computation gives

$$\Delta_M |\nabla_M x_{n+1}|^2/2 = -|Ae_{n+1}|^2 - (e_{n+1} \cdot \nu)\mathrm{div}_M(Ae_{n+1}).$$

We want now to find a more convenient expression for $\mathrm{div}_M(Ae_{n+1})$, in order to make a comparison with $|Ae_{n+1}|^2$ and establish the sign of $\Delta_M |\nabla_M x_{n+1}|^2/2$. Since $A\nu = 0$, $Ae_{n+1} = A\tau$, where $\tau := e_{n+1} - (e_{n+1} \cdot \nu)\nu$. Let $s : \Omega \to \mathbb{R}$ be the signed distance from M, and let us denote with an upper index the coordinates of τ. Using the Einstein convention for repeated inexes,

$$\begin{aligned}\mathrm{div}_M(A\tau) &= \partial_i (s_{ij}\tau^j) \\ &= \nabla(\Delta s) \cdot \tau + s_{ij}\tau^j_i \\ &= \nabla(\Delta s) \cdot \tau - (e_{n+1} \cdot \nu)s_{ij}\nu^j_i - (e_{n+1} \cdot \nu_i)s_{ij}\nu^j.\end{aligned}$$

Now, since $\tau \in T_M$, and $\Delta s = \mathrm{Tr}(A) \equiv 0$ on M, $\nabla(\Delta s) \cdot \tau = 0$. Moreover, $s_{ij}\nu^j = (A\nu)^i = 0$, and $\nu^j_i = s_{ij}$, so that

$$\Delta_M |\nabla_M x_{n+1}|^2/2 = -|Ae_{n+1}|^2 + (e_{n+1} \cdot \nu)^2 |A|^2. \tag{2.50}$$

Let us call $v := |\nabla_M x_{n+1}|^2/2$

If we knew that $\Delta_M |\nabla_M x_{n+1}|^2/2 \geq 0$ on M we could conclude the proof by the maximum principle, but we don't have a priori any control of $(e_{n+1} \cdot \nu)$

away fromn ∂M. Since our hypothesis is that $e_{n+1} \cdot \nu \neq 0$, we can define the smooth vector field on M by

$$b := -\frac{Ae_{n+1}}{(e_{n+1} \cdot \nu)},$$

so that $b \cdot \nabla_M v = |Ae_{n+1}|^2$. Thus, v is a strong solution of the equation

$$\Delta_M v + b \cdot \nabla_M v = (e_{n+1} \cdot \nu)^2 |A|^2 \geq 0, \tag{2.51}$$

and the maximum principle gives

$$\eta A \eta \geq 0 \qquad \text{at}\, \partial M.$$

□

The boundary inequality

Here we fix $x_0 \in \partial M \setminus \{0\}$, and we write Ω locally in x_0 as a graph of a positive function $u : \mathbb{R}^n \to [0, \infty)$, with $u(x_0) = 0$. Up to an isometry we can assume that $T_{\partial M}(x_0) = \mathbb{R}^{n-1}$, and thus

$$u_i(x_0) = 0, \qquad i < n.$$

As a consequence we have that $T_M(x_0) = \mathbb{R}^{n-1} \oplus \mathrm{Span}(e_n, u_n(x_0))$, and thus

$$\eta(x_0) = \frac{(e_n, u_n(x_0))}{\sqrt{1 + |\nabla u|^2(x_0)}}, \qquad \nu \circ \psi(x) = \frac{(-\nabla u(x), 1)}{\sqrt{1 + |\nabla u|^2(x_0)}}$$

where we are assuming also that $u_n(x_0) < 0$, and we are calling
$\psi(x) = (x, u(x))$. Since ∂M lies on $\mathbb{R}^n$, then ψ is the identity on ∂M. Thus, taking a path $\gamma(t)$ with values in ∂M,

$$u \circ \gamma \equiv 0.$$

Taking two derivatives of this condition we get

$$\begin{aligned}
&\nabla u_{\gamma(t)} \cdot \gamma'(t) = 0, \\
&\nabla u_{\gamma(t)} \cdot \gamma''(t) + \sum_{j=1}^{n} \nabla u_j \cdot \gamma'(t)(\gamma'(t) \cdot e_j) = 0.
\end{aligned}$$

Taking γ such that $\gamma(0) = x_0$ and $\gamma'(0) = e_i$, $i < n$, the second equation becomes

$$u_n \gamma''(0) \cdot e_n + u_{ii} = 0, \qquad i < n \tag{2.52}$$

The boundary condition $e_{n+1} \cdot \nu = \cos(\theta)$ is, for $x \in \partial M$,

$$|\nabla u|^2(x) = \tan^2(\theta). \tag{2.53}$$

We want to derive informiations taking derivatives of this condition. Take $\gamma(t)$ a path with values in ∂M, and taking first and second derivatives of (2.53) along γ we get:

$$\sum_{j=1}^{n} u_j \nabla u_j \cdot \gamma'(t) = 0, \tag{2.54}$$

$$\sum_{j=1}^{n} |\nabla u_j \cdot \gamma'(t)|^2 + u_j \nabla u_j \cdot \gamma''(t) + \sum_{j=1}^{n}\sum_{k=1}^{n} u_j \nabla u_{jk} \cdot \gamma'(t)(\gamma'(t))^k = 0, \tag{2.55}$$

where we indicate with the upper idexes the coordinates of γ'. Evaluating the first equation at x_0 we can write,

$$u_{ni}(x_0) = 0, \qquad i < n, \tag{2.56}$$

and by that we can also, up to an isometry, assume that $\nabla^2 u(x_0)$ is diagonal, and that u is 0-homogeneous at x_0 in direction e_1. By that, we can take $\gamma'(x_0) = e_i$ with $i < n$, and we can evaluate in x_0 the second equation in (2.54), getting

$$u_{ii}^2 + u_n u_{nn} e_n \cdot \gamma''(0) + u_n u_{nii} = 0,$$

that is, by (2.52),

$$u_{ii}^2 - u_{ii}u_{nn} + u_n u_{nii} = 0, \qquad i < n. \tag{2.57}$$

Like in the proof of Lemma 1.3.8, we can write

$$-A \circ \psi(x) = \frac{\nabla^2 u(x)}{\sqrt{1 + |\nabla u|^2(x)}} g^{-1},$$

thus

$$|A|^2(x_0) = (1 + \tan^2(\theta))^{-1} \left(\sum_{i<n} u_{ii}^2 + \frac{u_{nn}^2}{(1 + \tan^2(\theta))^2} \right) \tag{2.58}$$

, and the zero mean curvature condition on M at x_0 reads as

$$\sum_{i<n} u_{ii} = -\frac{u_{nn}}{1 + \tan^2(\theta)}. \tag{2.59}$$

Notice that, if $i < n$, $\mathrm{Tr}(\nabla u_i \otimes \nabla u) = \sum_{j=1}^{n} u_{ij} u_j = u_{in} u_n = 0$, and in the same way $\partial_i |\nabla u|^{2m} = 0$, for any natural number m.

For the derivative in direction η at x_0,

$$0 = \partial_n(Id) = \partial_n(gg^{-1}) = \partial_n g g^{-1} + g\partial_n(g^{-1}),$$

thus

$$\partial_n(g^{-1}) = -g^{-1}\partial_n(g)g^{-1}.$$

Taking into account that $\nabla u_n \otimes \nabla u(x_0)$ is dyagonal, and that, in our coordinates, its only non zero entrance is the one in position (n, n), and that $g^{-1}(x_0)$ has $(1 + \tan^2(\theta))^{-1}$ as (n, n) entrance, we have

$$\partial_n g^{-1}(x_0) = -2g^{-1}\nabla u_n \otimes \nabla u g^{-1} = -\frac{2}{(1 + \tan^2(\theta))^2}\nabla u_n \otimes \nabla u,$$

thus,

$$\begin{aligned} -\partial_n(A \circ \psi) = \frac{\nabla^2 u_n}{\sqrt{1 + \tan^2(\theta)}} g^{-1} + u_{nn} \tan(\theta) \frac{\nabla^2 u}{(1 + \tan^2(\theta))^{3/2}} g^{-1} \\ - 2\frac{\nabla^2 u}{(1 + \tan^2(\theta))^{5/2}} \nabla u_n \otimes \nabla u, \end{aligned} \tag{2.60}$$

and the condition $\mathrm{Tr}(\partial_n(A \circ \psi)) = 0$ reads as

$$\sum_{i<n} u_{iin} + \frac{u_{nnn}}{1 + \tan^2(\theta)} + 2u_{nn}^2 \frac{\tan(\theta)}{(1 + \tan^2(\theta))^2} = 0. \tag{2.61}$$

Notice also that, in our coordinate system,

$$\eta \cdot A\eta(x_0) = -\frac{u_{nn}}{(1 + \tan^2(\theta))^{3/2}}. \tag{2.62}$$

Here we are abusing in the notation, since, during the proof, we referred at A as an $n \times n$ matrix in our coordinates, but we recall that A can also be thought as an $(n + 1) \times (n + 1)$-symmetric matrix that is zero in the normal direction to M.

We want to find a manageable expression for the boundary term $\partial_\eta |A|^2$. By the choice of our coordinates,

$$\partial_\eta |A|^2(x_0) = \frac{\partial_n(|A|^2 \circ \psi)}{\sqrt{1 + \tan^2(\theta)}}(x_0),$$

and

$$
\begin{aligned}
(1+\tan^2(\theta))\partial_n\left(\frac{|A|^2\circ\psi}{2}\right)(x_0) &= (1+\tan^2(\theta))\mathrm{Tr}(\partial_n(A\circ\psi)A\circ\psi)\\
&= \sum_{i<n} u_{iin}u_{ii} + \frac{u_{nnn}u_{nn}}{(1+\tan^2(\theta))^2}\\
&+ u_{nn}\frac{\tan(\theta)}{1+\tan^2(\theta)}\left(\sum_{i<n} u_{ii}^2 + \frac{u_{nn}^2}{(1+\tan^2(\theta))^2}\right)\\
&+ 2u_{nn}^3\frac{\tan(\theta)}{(1+\tan^2(\theta))^3}.
\end{aligned}
$$

We want now to use all our previous equations in the case in which the dimension is $n=3$. By homogeneity in the e_1 direction,

$$u_{11} = u_{113} = 0.$$

The equation (2.57), together with (2.59), gives an expression of u_{223} in terms of u_{33}:

$$
\begin{aligned}
u_{223} &= \frac{u_{22}^2}{\tan(\theta)} - \frac{1}{\tan(\theta)}u_{33}u_{22}\\
&= u_{33}^2\left(\frac{1}{\tan(\theta)(1+\tan^2(\theta))^2} + \frac{1}{\tan(\theta)(1+\tan^2(\theta))}\right)\\
&= u_{33}^2\frac{2+\tan^2(\theta)}{\tan(\theta)(1+\tan^2(\theta))^2},
\end{aligned}
$$

and by (2.61) we can write u_{333} in terms of u_{33}:

$$
\begin{aligned}
\frac{u_{333}}{1+\tan^2(\theta)} &= -u_{223} - 2u_{33}^2\frac{\tan(\theta)}{(1+\tan^2(\theta))^2}\\
&= -u_{33}^2\frac{1}{\tan(\theta)(1+\tan^2(\theta))^2}\left(2+3\tan^2(\theta)\right).
\end{aligned}
$$

Putting everything together, and using again $u_{22}^2 = \frac{u_{33}^2}{(1+\tan^2(\theta))^2}$, and (2.58), we can write

$$
\begin{aligned}
(1+\tan^2(\theta))\partial_3\left(\frac{|A|^2\circ\psi}{2}\right)(x_0) &= -u_{33}^3\frac{2+\tan^2(\theta)}{\tan(\theta)(1+\tan^2(\theta))^3}\\
&- u_{33}^3\frac{2+3\tan^2(\theta)}{\tan(\theta)(1+\tan^2(\theta))^3}
\end{aligned}
$$

$$\begin{aligned}
&+ 2u_{33}|A|^2 \tan(\theta) \\
&= -4u_{33}^3 \frac{1+\tan^2(\theta)}{\tan(\theta)(1+\tan^2(\theta))^3} \\
&\quad + 2u_{33}|A|^2 \tan(\theta) \\
&= -2u_{33}|A|^2 \frac{1}{\tan(\theta)}
\end{aligned}$$

Since $u_{33} = -\eta \cdot A\eta(1+\tan^2(\theta))^{3/2}$, we can summarize the previous discussion in the following theorem.

Theorem 2.5.4. *If Ω is a smooth cone, and $n = 3$, then*

$$\partial_\eta \frac{|A|^2}{2} = 2\cot(\theta)\eta \cdot A\eta|A|^2, \qquad \text{at } \partial M \tag{2.63}$$

Corollary 2.5.5. *Let $n = 3$. Then $c := |A|^{\frac{1}{2}}$ satisfies the hypothesis of Proposition 2.5.1.*

In particular, $|A| \equiv 0$.

Proof. Let $\alpha = \frac{1}{2}$. We just need to check that $c = |A|^\alpha$ satisfies

$$\cot(\theta)|A|^\alpha \eta \cdot A\eta - \partial_\eta |A|^\alpha \geq 0, \qquad \text{at } \partial M.$$

By (2.63),

$$\partial_\eta |A|^\alpha = \alpha |A|^{\alpha-2} \partial_\eta \frac{|A|^2}{2} = 2\alpha \cot(\theta)\eta \cdot A\eta |A|^\alpha.$$

Therefore,

$$\cot(\theta)|A|^\alpha \eta \cdot A\eta - \partial_\eta |A|^\alpha = \Big(\cot(\theta)|A|^\alpha \eta \cdot A\eta\Big)(1-2\alpha) = 0,$$

since $\alpha = \frac{1}{2}$. ∎

Let use generalize the computations to any dimension. For any n we have the formula

$$\begin{aligned}
(1+\tan^2(\theta))\partial_n\left(\frac{|A|^2 \circ \psi}{2}\right)(x_0) &= (1+\tan^2(\theta))\mathrm{Tr}(\partial_n(A\circ\psi)A\circ\psi) \\
&= \sum_{i<n} u_{iin}u_{ii} + \frac{u_{nnn}u_{nn}}{(1+\tan^2(\theta))^2} \\
&\quad + u_{nn}\frac{\tan(\theta)}{1+\tan^2(\theta)}\left(\sum_{i<n} u_{ii}^2 + \frac{u_{nn}^2}{(1+\tan^2(\theta))^2}\right)
\end{aligned}$$

$$+ 2u_{nn}^3 \frac{\tan(\theta)}{(1+\tan^2(\theta))^3}.$$

Now

$$\begin{aligned}\sum_{i<n} u_{iin}u_{ii} &= -\frac{1}{u_n}\sum_{i<n} u_{ii}^3 + \frac{u_{nn}}{u_n}\sum_{i<n} u_{ii}^2 \\ &= \cot(\theta)\sum_{i<n} u_{ii}^3 - \cot(\theta)u_{nn}\sum_{i<n} u_{ii}^2,\end{aligned}$$

while

$$\begin{aligned}\frac{u_{nnn}u_{nn}}{(1+\tan^2(\theta))^2} =& - 2u_{nn}^3 \frac{\tan(\theta)}{(1+\tan^2(\theta))^3} \\ & - \sum_{i<n} \frac{u_{iin}u_{nn}}{1+\tan^2(\theta)} \\ =& - 2u_{nn}^3 \frac{\tan(\theta)}{(1+\tan^2(\theta))^3} \\ & + \frac{u_{nn}^2}{\tan(\theta)(1+\tan^2(\theta))}\sum_{i<n} u_{ii} \\ & - \frac{u_{nn}}{\tan(\theta)(1+\tan^2(\theta))}\sum_{i<n} u_{ii}^2 \\ =& -2u_{nn}^3 \frac{\tan(\theta)}{(1+\tan^2(\theta))^3} \\ & - \frac{u_{nn}^2}{(1+\tan^2(\theta))}\frac{u_{nn}}{\tan(\theta)(1+\tan^2(\theta))} \\ & - \frac{u_{nn}}{\tan(\theta)(1+\tan^2(\theta))}\sum_{i<n} u_{ii}^2 \\ =& - 2u_{nn}^3 \frac{\tan(\theta)}{(1+\tan^2(\theta))^3}\end{aligned}$$

Thus,

$$\begin{aligned}(1+\tan^2(\theta))\partial_n(\frac{|A|^2}{2}) =& \cot(\theta)\left(\sum_{i<n} u_{ii}^3 + \frac{u_{nn}^3}{(1+\tan^2(\theta))^3}\right) \\ & - \cot(\theta)u_{nn}\sum_{i<n} u_{ii}^2 \\ & - \cot(\theta)\frac{u_{nn}^3(2+\tan^2(\theta))}{(1+\tan^2(\theta))^3}\end{aligned}$$

$$
\begin{aligned}
&- \cot(\theta)\frac{u_{nn}}{1+\tan^2(\theta)}\sum_{i<n} u_{ii}^2 \\
&+ \cot(\theta)u_{nn}\tan^2(\theta)|A|^2 \\
= \cot(\theta)&\left(\sum_{i<n} u_{ii}^3 + \frac{u_{nn}^3}{(1+\tan^2(\theta))^3}\right) \\
&- \cot(\theta)u_{nn}\frac{2+\tan^2(\theta)}{1+\tan^2(\theta)}\sum_{i<n} u_{ii}^2 \\
&- \cot(\theta)u_{nn}\frac{2+\tan^2(\theta)}{1+\tan^2(\theta)}\frac{u_{nn}^2}{(1+\tan^2(\theta))^2} \\
&+ \cot(\theta)u_{nn}\tan^2(\theta)|A|^2 \\
= \cot(\theta)&\left(\sum_{i<n} u_{ii}^3 + \frac{u_{nn}^3}{(1+\tan^2(\theta))^3}\right) \\
&- \cot(\theta)u_{nn}(2+\tan^2(\theta))A|^2 \\
&+ \cot(\theta)u_{nn}\tan^2(\theta)|A|^2 \\
= \cot(\theta)&\left(\sum_{i<n} u_{ii}^3 + \frac{u_{nn}^3}{(1+\tan^2(\theta))^3} - 2u_{nn}|A|^2\right).
\end{aligned}
$$

This proves that

Theorem 2.5.6.

$$
\partial_\eta \frac{|A|^2}{2} = \cot(\theta)\left(2\eta\cdot A\eta|A|^2 - \sum_{i=1}^{n}\lambda_i^3\right), \tag{2.64}
$$

where $(\lambda_i)_{i=1}^n$ are the eigenvalues of A.

Under the assumptions of Lemma 2.5.3 notice that now we have

- The instability crytherion given by Proposition 2.5.1.
- The boundary curvature term $\eta \cdot A\eta \geq 0$.
- An expression of $\frac{1}{\cot(\theta)}\partial\eta|A|^2$ in terms of $\eta\cdot A\eta$, $|A|^2$ and the sum of cubes of the eigenvalues of A.
- The Simons' inequality.

Making a comparison with [3], the next step would be to take the right function of the eigenvalues of A as a competitor in Proposition 2.5.1. We had no time to find the right competitor and proving that it works. However, we leave here our candidate as a conjecture.

Conjecture 2.5.7. *Let $n = 4$, and let*

$$w^2 := 4 \sum_{\lambda_k > 0} \lambda_k^2 + \sum_{\lambda_k < 0} \lambda_k^2,$$

Where λ_k are the eigenvalues of A.

Then, if the assumptions of Lemma 2.5.3 are satisfied, $c := w^{\frac{1}{3}}$ satisfies the hypothesis of Proposition 2.5.1.

In particular, $|A| \equiv 0$.

2.6 The case $4 \le n \le 6$ with θ close to $\frac{\pi}{2}$

When the angle θ is close to $\frac{\pi}{2}$, it is just sufficient to choose the right competitor in the Stability inequality, like in the proof of Simons' Theorem.

We first introduce a slightly improved version of the Simons' inequality.

Lemma 2.6.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open cone with an isolated singularity at 0, such that $\partial\Omega := M$ has zero mean curvature. Thenm for any $\lambda \in (0, 1)$,*

$$\Delta_M \frac{|A|^2}{2} + |A|^4 \ge \lambda\left(2\frac{|A|^2}{|x|^2} + |\nabla_M |A||^2\right) + (1-\lambda)\left(1 + \frac{2}{n}\right)|\nabla_M |A||^2. \quad (2.65)$$

Proof. Thanks to the Simons' inequality, it is sufficient to prove that

$$\Delta_M \frac{|A|^2}{2} + |A|^4 \ge \left(1 + \frac{2}{n}\right)|\nabla_M |A||^2.$$

Fix $p \in M$, and take coordinates around p given by u, like in the proof of Theorem 1.3.9. Then, we have at p,

$$\Delta_M \frac{|A|^2}{2} + |A|^4 = \sum_{ijk=1}^{n} u_{ijk}^2,$$

$$A = \nabla^2 u$$

and

$$|A|^2 |\nabla_M |A||^2 = \sum_{i=1}^{n}\left(\sum_{j,k=1}^{n} u_{ijk} u_{jk}\right)^2.$$

Thus, by the Cauchy-Schwartz inequality, and since $\nabla^2 u$ is dyagonal at p,

$$|\nabla_M |A||^2 \le \sum_{ij=1}^{n} u_{iij}^2. \quad (2.66)$$

Since the mean curvature of M is zero, so is its first derivative, so, by Lemma 1.3.8,

$$u_{iii} = -\sum_{j \neq i} u_{jji}.$$

Thanks to the Cauchy-Schwartz inequality, for any $b_1, \cdot b_n$ real numbers,

$$\left(\sum_{i=1}^{n-1} b_i\right)^2 \leq (n-1) \sum_{i=1}^{n-1} b_i^2,$$

that, together with the previous estimates, gives,

$$\begin{aligned}
|\nabla_M |A||^2 &\leq \sum_{i \neq j} u_{iij}^2 + \sum_{i=1}^{n} \left(\sum_{j \neq i} u_{iij}\right)^2 \\
&\leq \sum_{i \neq j} u_{iij}^2 + (n-1) \sum_{i=1}^{n} \sum_{j \neq i} u_{iij}^2 \\
&= n \sum_{i \neq j} u_{iij}^2.
\end{aligned}$$

Combining this equation with (2.66), we get

$$\left(1 + \frac{2}{n}\right) |\nabla_M |A||^2 \leq \sum_{i,j,k=1}^{n} u_{ijk}^2,$$

that ends the proof. $\square$

We need now a trace inequality involving the contact angle.

Lemma 2.6.2. *Let $\Omega \subset \mathbb{R}^{n+1}_+$ an open set with $M := \partial\Omega \cap \mathbb{R}^{n+1}_+$ smooth up to the boundary, meeting $\mathbb{R}^n$ at a constant angle θ. Then, for any $u \in C^\infty_c(M)$, we have*

$$\int_{\partial M} u \leq \frac{1}{\sin(\theta)} \int_M |\nabla u| \tag{2.67}$$

Proof. Consider the vector field $\xi(x) := -\varphi_R(x_{n+1}) e_{n+1}$, where $\varphi_R : [0, \infty) \in [0, 1)$ is a smooth function such that $\varphi_R = 1$ on $[0, R]$, $\varphi_R = 0$ on $[2R, \infty]$, and $|\varphi_R'| \leq \frac{2}{R}$. Since $\xi \cdot \eta 0 \sin(\theta)$ at ∂M, thanks to the divergence theorem,

$$\int_{\partial M} u = \frac{1}{\sin(\theta)} \int_{\partial M} u \xi \cdot \eta$$

$$
\begin{aligned}
&= \frac{1}{\sin(\theta)} \int_M \mathrm{div}_M(u\xi) \\
&\leq \frac{1}{\sin(\theta)} \int_M |\nabla_M u||\xi| + u\,\mathrm{div}_M \xi \\
&\leq \frac{1}{\sin(\theta)} \int_M |\nabla_M u| + 2\frac{u}{R}.
\end{aligned}
$$

Taking the limit as $R \to \infty$ we obtain the thesis. □

Remark 2.6.3. In the next proof, we apply the stability inequality, the integration by parts, and the trace inequality for functions not smooth and compactly supported in $\mathbb{R}^{n+1} \setminus \{0\}$, but an approximation argument makes the proof below rigorous.

Proof of Theorem 2.2.3 when θ is close to $\frac{\pi}{2}$. .
Let $p \in (\frac{1}{2}, 1)$ to be chosen later. For any $\lambda \in (0,1)$, by (2.65), we can wrie

$$
\begin{aligned}
\frac{\Delta_M |A|^{2p}}{2} &= \frac{p}{2}\mathrm{div}(|A|^{2p-2}\nabla_M |A|^2) \\
&= p\frac{\Delta_M |A|^2}{2}|A|^{2p-2} + p(2p-2)|A|^{2p-2}|\nabla_M |A||^2 \\
&\geq p|A|^{2p-2}|\nabla_M |A||^2\left((1-\lambda)\left(1+\frac{2}{n}\right) + \lambda + 2p - 2\right) \\
&\qquad + p|A|^{2p}\left(\frac{2\lambda}{|x|^2} - |A|^2\right). \qquad (2.68)
\end{aligned}
$$

Let $r = |x|$, and take $f = f(r)$ to be a radial Lipschitz function compactly supported in $\mathbb{R}^{n+1} \setminus \{0\}$. Plugging $\varphi = f|A|^{2p}$ into the stability inequality (2.21) and integrating by parts, we obtain

$$
\begin{aligned}
\cot(\theta)\int_{\partial M} \eta \cdot A\eta |A|^{2p} f^2 + \int_M |A|^{2p+2} f^2 &\leq \int_M |\nabla_M (f|A|^p)|^2 \\
= \int_M p^2 |A|^{2p-2}|\nabla_M |A||^2 f^2 &+ |A|^{2p}|\nabla_M f|^2 + \frac{1}{2}\nabla_M |A|^{2p} \cdot \nabla_M f^2 \\
= \int_M p^2 |A|^{2p-2}&|\nabla_M |A||^2 f^2 + |A|^{2p}|\nabla_M f|^2 \\
&- \int_M \frac{f^2}{2}\Delta_M |A|^{2p} + \int_{\partial M} \frac{f^2}{2}\partial_\eta |A|^{2p}. \qquad (2.69)
\end{aligned}
$$

By (2.64), we have

$$\partial_\eta \frac{|A|^2}{2} = \cot(\theta)\bigg(\sum_{i=1}^{n} \lambda_i^3 - 2\eta \cdot A\eta |A|^2\bigg),$$

where λ_i are the eigenvalues of A. Notice now that, for any $i = 1 \to n$, by the Cauchy-Schwartz inequality,

$$\begin{aligned}
|\lambda_i| &= \bigg|\sum_{j\neq i} \lambda_j\bigg| \\
&\leq \sqrt{n-1}\bigg(\sum_{j\neq i} \lambda_i^2\bigg)^{1/2} \\
&\leq \sqrt{n-1}|A|,
\end{aligned}$$

thus

$$\bigg|\sum_{i=1}^{n} \lambda_i^3\bigg| \leq \sqrt{n-1}\sum_{i=1}^{n} \lambda_i^2 |A| = \sqrt{n-1}|A|^3.$$

From this, we get

$$\bigg|\partial_\eta \frac{|A|^2}{2}\bigg| \leq 3\cot(\theta)\sqrt{n-1}|A|^3.$$

Using this bound and (2.67), we can estimate

$$\begin{aligned}
&\bigg|\cot(\theta)\int_{\partial M} \eta \cdot A\eta |A|^{2p} f^2 - \int_{\partial M} \frac{f^2}{2}\partial_\eta |A|^{2p}\bigg| \leq c(n)\cot(\theta)\int_{\partial M} f^2 |A|^{2p+1} \\
&\leq c(n)\frac{\cot(\theta)}{\sin(\theta)}\int_M |\nabla_M (f^2|A|^{2p+1})| \\
&\leq c(n)\frac{\cot(\theta)}{\sin(\theta)}\int_M (2p+1)|\nabla_M |A|||A|^{2p} f^2 + 2|f||\nabla_M f||A|^{2p+1} \\
&\leq c(n)\frac{\cot(\theta)}{\sin(\theta)}\int_M 3|\nabla_M |A|||A|^{-1/2}|A|^{1/2}|A|^{2p} f^2 + 2|f||\nabla_M f||A|^{2p+1} \\
&\leq c(n)\frac{\cot(\theta)}{\sin(\theta)}\int_M 3|\nabla_M |A||^2 |A|^{2p-2} f^2 + f^2|A|^{2p+2} + 2|f||A|^p|\nabla_M f||A|^{p+1} \\
&\leq c(n)\frac{\cot(\theta)}{\sin(\theta)}\int_M 3|\nabla_M |A||^2 |A|^{2p-2} f^2 + 2f^2|A|^{2p+2} + |\nabla_M f|^2|A|^{2p},
\end{aligned}$$

where $c(n)$ is a positive constant depending only on n. We used in the computations above that $3ab \leq 3a^2 + b^2$, $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, and that $2p+1 < 3$. Plugging this into (2.69), and using (2.68), we get

$$0 \leq \int\limits_M |A|^{2p-2}|\nabla_M|A||^2 f^2 \left(c(n)\frac{\cot(\theta)}{\sin(\theta)} + p\left(-p + 2 - (1-\lambda)(1+\frac{2}{n}) - \lambda \right) \right)$$
$$+ \int\limits_M |\nabla_M f|^2 |A|^{2p}(1 + c(n)\frac{\cot(\theta)}{\sin(\theta)}) - 2p\lambda \frac{|A|^{2p}}{|x|^2} f^2$$
$$+ \int\limits_M |A|^{2p+2} f^2 (p + -1 + 2c(n)\frac{\cot(\theta)}{\sin(\theta)}) \quad (2.70)$$

set $\epsilon > 0$, and define the radial lipschitz function $f(r)$ by

$$f(r) = \begin{cases} r^{1+\epsilon}, & r \leq 1 \\ r^{2-n/2-\epsilon}. \end{cases}$$

f is not compactly supported, but $\int\limits_0^\infty r^{n-2} f(r)\, dr < \infty$, and thus the right hand side of (2.69) is finite with this choice of f.

Like in the proof of Theorem 1.3.10, now is just a matter of computation. It can be seen that, if $\theta \in (\theta_1(n), \frac{\pi}{2})$ is close enough to $\frac{\pi}{2}$, we can choose parameter p, ϵ, λ such that, when $n \leq 6$, (2.70) is satisfied if and only if $|A| = 0$. □

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