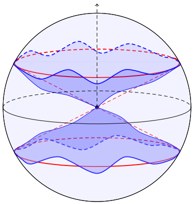
On the logarithmic epiperimetric inequality



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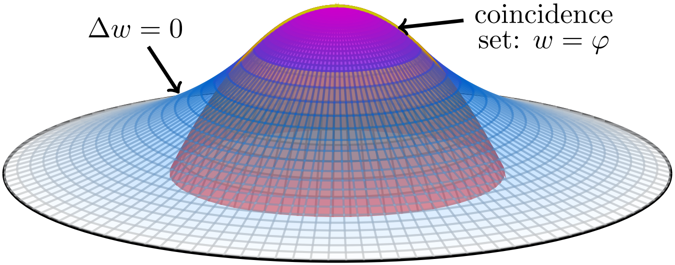
Given: - *a domain* $D \subseteq \mathbb{R}^d$,
- *a function* $\varphi : D \to \mathbb{R}$,
- *a boundary datum* $f : \partial D \to \mathbb{R}$.

φ is the obstacle

$f \geq \varphi$ on ∂D



Minimize $\int_D |\nabla w|^2 \, dx$ **among** *all the functions* $w : D \to \mathbb{R}$ **such that** $w = f$ *on* ∂D **and** $w \geq \varphi$ *in* D.

Minimize $\displaystyle\int_D |\nabla w|^2\,dx$ **among** *all the functions* $w : D \to \mathbb{R}$
such that $w = f$ *on* ∂D **and** $w \geq \varphi$ *in* D.

Setting:
$u := w - \varphi$
$g := f - \varphi$

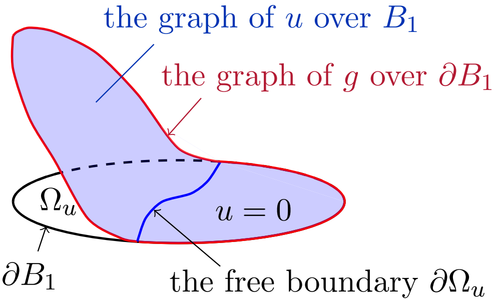
Standart assumtion:
$\Delta\varphi = 1/2$

$$\begin{aligned}\int_D |\nabla w|^2\,dx &= \int_D |\nabla(u+\varphi)|^2\,dx\\ &= \int_D |\nabla u|^2\,dx + \int_D 2\nabla\varphi\cdot\nabla u\,dx + \int_D |\nabla\varphi|^2\,dx\\ &= \int_D |\nabla u|^2\,dx + \int_D (2\Delta\varphi)\,u\,dx + \text{const.}\end{aligned}$$

Minimize $\displaystyle\int_D \left(|\nabla u|^2 + u\right)dx$ **among** *all the functions* $u : D \to \mathbb{R}$
such that $u = g$ on ∂D **and** $u \geq 0$ in D.

1968 Brezis-Stampacchia (*Bull. Soc. Math. Fr.*) - $u \in C^{1,\alpha}$

Corollary:
- $\Omega_u = \{u > 0\}$ is open
- $\Delta u = 1/2$ in Ω_u
- ∇u is defined on $\partial\Omega_u$.



Optimality condition:

$$|\nabla u| = 0 \quad \text{on} \quad \partial\Omega_u$$

$$\Delta u = \frac{1}{2}\mathbb{1}_{\{u>0\}} \quad \text{in} \quad B_1$$

1973 Gerhardt (*Arch. Rat. Mech. Anal.*) - $u \in C^{1,1}$

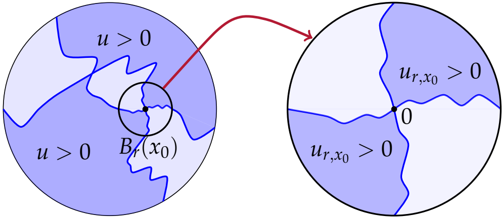
Blow-up:

For $x_0 \in \partial\Omega_u$ and $r > 0$, define

$$u_{r,x_0}(x) := \frac{1}{r^2} u(x_0 + rx)$$

Then: • u_{r,x_0} is a solution in B_1;

• $|\partial_i \partial_j u_{r,x_0}| \leq C$.



Corollary (Compactness of the blow-up sequences). Let $r_n \to 0$ and $x_0 \in \partial\Omega_u$. Then, up to a subsequence,

$$u_{r_n,x_0}(x) := \frac{1}{r_n^2} u(x_0 + r_n x),$$

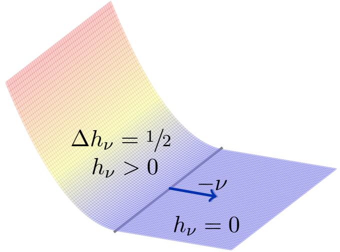
converges to a **blow-up limit** u_0.

Regular blow-up limits.

The blow-up limit $u_0 : B_1 \to \mathbb{R}$ is regular if there is a vector $\nu \in \partial B_1$ such that $u_0 = h_\nu$.

Half-plane solutions:

$$h_\nu(x) = \frac{1}{2}(x \cdot \nu)_+^2$$

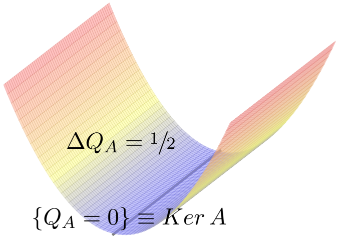


Singular blow-up limits.

The blow-up limit $u_0 : B_1 \to \mathbb{R}$ is singular if there is a matrix A such that $u_0 = Q_A$.

Global singular solutions:

$$Q_A(x) = \frac{1}{2} x \cdot Ax \quad \text{where} \quad tr\, A = \frac{1}{4}.$$

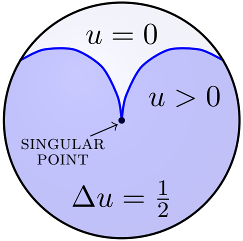


Theorem (Caffarelli)**:**
The free boundary can be decomposed as

$$\partial\Omega_u = Reg(\partial\Omega_u) \cup Sing(\partial\Omega_u)$$

$$Reg(\partial\Omega_u) := \Big\{x_0 \in \partial\Omega_u \ : \ \text{every blow-up at } x_0 \text{ is } \textbf{regular}\Big\}$$

$$Sing(\partial\Omega_u) := \Big\{x_0 \in \partial\Omega_u \ : \ \text{every blow-up at } x_0 \text{ is } \textbf{singular}\Big\}$$



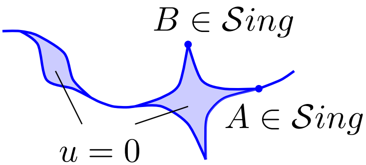
Structure of the regular part of the free boundary

1977 Caffarelli (*Acta Math.*) - *$Reg(\partial\Omega_u)$ is a $C^{1,\alpha}$-regular manifold.*

1977 Kinderlehrer-Nirenberg (*Ann. Sc. Norm. Sup. Pisa*) - $C^{1,\alpha}$ => *analytic.*

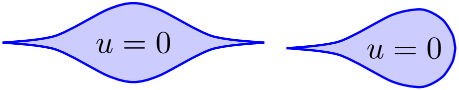
What about the (closed) singular set $Sing(\partial\Omega_u)$?

1977 Caffarelli-Riviere (*Ann. of Math.*)



2003 Monneau (*J. Geom. Anal.*)

Monneau monotonicity formula

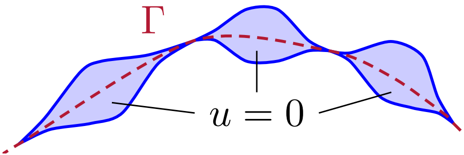


1976 Schaeffer (*Ann. SNS Pisa*)

1991 Sakai (*Acta Math.*)

1996 Sakai (*Ann. SNS Pisa*)

1999 Weiss (*Invent. Math.*)



- $\Gamma \in C^1$ (Caffarelli-Riviere, Monneau); $\Gamma \in C^{1,\alpha}$ (Weiss);
- $\Gamma \cap Sing$ *might be a Cantor set if* $\Delta u = f(x)\mathbb{1}_{\{u>0\}}$ (Schaeffer);
- $\Gamma \cap Sing$ *is finite if* $f(x)$ *is analytic* (Sakai).

Stratification of the singular set

- At every $x_0 \in Sing$ the blow-up is **unique** (Caffarelli'98):

$$u_{x_0} = Q_{A_{x_0}} = x \cdot A_{x_0}[x]$$

- Every $x_0 \in Sing$ has a **rank**:

$$\mathrm{Rank}\,(x_0) = \dim \mathrm{Ker}\, A_{x_0}.$$

- We define the m-th **stratum** Σ_m of the singular set as:

$$x_0 \in \Sigma_m \Leftrightarrow \mathrm{Rank}\,(x_0) = m.$$

1998 Caffarelli (*J. Fourier Anal. Appl.*): $\Sigma_m \in C^1$

2001 Monneau (*Progr. Math.*): $\Sigma_m \in C^1$

2018 Colombo-Spolaor-Velichkov (*Geom. Funct. Anal.*): $\Sigma_m \in C^{1,\log}$

2018 Figalli-Serra (*Invent. Math.*): $C^{1,\log}$ is optimal, but Σ_m is a.e. $C^{1,1}$ or better.

Minimal surfaces

1960 Reifenberg (*Ann. of Math.*) - *regularity;*

1966 Jean Taylor (*Ann. of Math.*) - Y*-type singularities;*

1984 Brian White (*Duke Math. J.*) - *two-dimensional currents.*

The Weiss approach to the regularity of the flat free boundaries

1999 Weiss (*Invent. Math.*) - *obstacle problem;*

2016 Focardi-Spadaro (*Adv. Diff. Eq.*) - *thin-obstacle problem;*

2016 Garofalo-Petrosyan-Vega Garcia (*J. Math. Pures Appl.*) - *thin-obstacle problem.*

Direct approach –> log-epiperimetric inequality –> constructive approach

2017 Spolaor-Velichkov (*Comm. Pure Appl. Math.*) - *the one-phase problem;*

2017 Colombo-Spolaor-Velichkov (*Geom. Funct. Anal.*) - *obstacle problem;*

2017 Colombo-Spolaor-Velichkov (*Comm. Pure Appl. Math.*) - *thin-obstacle problem;*

2018 Engelstein-Spolaor-Velichkov - *the one-phase problem;*

2018 Engelstein-Spolaor-Velichkov (*Geom. Topol.*) - *(almost-)minimal surfaces;*

2018 Spolaor-Trey-Velichkov - *the two-phase problem (for almost-minimizers).*

Hypotheses

1. Monotonicity formula:

$$\frac{\partial}{\partial r}E(u_r) = \frac{1}{r}\big(E(z_r) - E(u_r)\big) + \frac{1}{r}\int_{\partial B_1} |x \cdot \nabla u_r - 2u_r|^2$$

2. Epiperimetric inequality: $E(u_r) \leq (1-\varepsilon)E(z_r)$

Thesis

There is a unique blow-up limit $u_0 = \lim_{r\to 0} u_r$

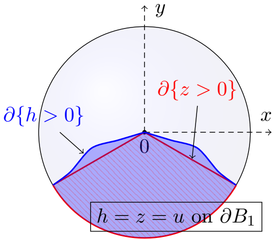
$\|u_r - u_0\|_{L^2(\partial B_1)} \leq r^\varepsilon$ *for every* $r > 0$

The free boundary is $C^{1,\varepsilon}$*-regular.*

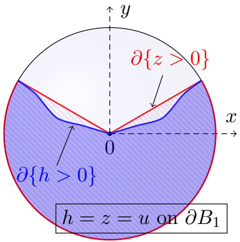
How to prove the epiperimetric inequality for the energy E?

Given *a 2-homogeneous nonnegative function* $z : B_1 \to \mathbb{R}$ *,*
find $h : B_1 \to \mathbb{R}$ *such that:*

$h \geq 0$ in B_1 | $h = z$ on ∂B_1 | $E(h) \leq (1-\varepsilon)E(z)$



Construction
of the competitior at
the flat (regular) points.
Spolaor-Velichkov
(CPAM 2017)



What if the point is singular?
At general singular points the epiperimetric inequality cannot hold!
Reifenberg (Ann. of Math. 1960) + *Figalli-Serra* (Invent. Math. 2018).

Theorem *(Colombo-Spolaor-Velichkov)*: *Let E be the obstacle-problem energy.*
Given *a 2-homogeneous nonnegative function* $z : B_1 \to \mathbb{R}$,
there exists $h : B_1 \to \mathbb{R}$ *such that:*

$h \geq 0$ in B_1

$h = z$ on ∂B_1

$$E(h) \leq \big(1 - |E(z)|^\gamma\big)E(z)$$

Corollary *(Colombo-Spolaor-Velichkov, GAFA 2018)*:
Let $u : B_1 \to \mathbb{R}$ be a solution to the obstacle problem.
Then the singular set is contained in a $C^{1,\log^\gamma}$-regular manifold.