Boundary regularity of the free interface in spectral optimal partition problems
by *Roberto Ognibene, Bozhidar Velichkov*

Cover letter

Setting. Fixed a C^1 bounded open set $D \subset \mathbb{R}^d$ and $N \geq 1$, we consider the optimal partition problem

$$\inf\left\{\sum_{i=1}^{N} \lambda_1(\Omega_i)\colon \Omega_i \subset D - \text{ open and such that } \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j\right\}, \qquad \text{(OP)}$$

where $\lambda_1(\Omega_i)$ denotes the first Dirichlet eigenvalue on Ω_i. This type of optimal partition problems has been studied in several frameworks such as dynamics of populations and harmonic maps with values in singular spaces. The regularity of the free interface

$$\mathcal{F} := \bigcup_{i=1}^{N} \partial\Omega_i \cap D, \quad \text{with } (\Omega_1, \dots, \Omega_N) \text{ being a minimizer of (OP)},$$

has been extensively studied in the interior of D. We refer for instance to [Alp, CL1, CL2, CTV1, CTV2], where it was proved that the interior interface $\mathcal{F}$ can be decomposed into the disjoint union of a smooth $(d-1)$-dimensional manifold and a closed $(d-2)$-rectifiable set.

Main results. In the present paper we investigate:

- the behavior of $\mathcal{F}$ up to the fixed boundary ∂D;
- the regularity of $\mathcal{F}_{\partial D} := \overline{\mathcal{F}} \cap \partial D$.

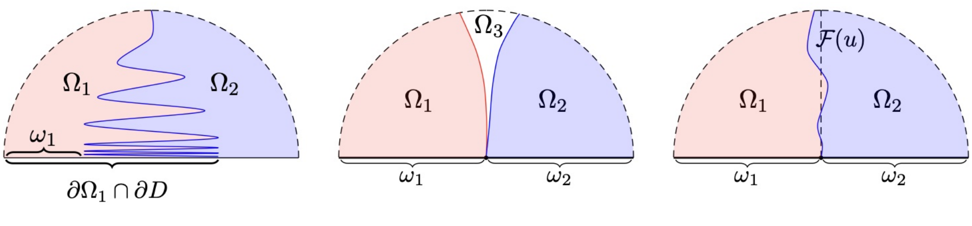
We introduce a family $(\omega_1, \dots, \omega_N)$ of relatively open subsets of ∂D such that:

$$\omega_i \cap \omega_j = \emptyset \text{ for } i \neq j\ , \qquad \bigcup_{i=1}^{N} \overline{\omega_i} = \partial D \qquad \text{and} \qquad \mathcal{F}_{\partial D} = \bigcup_{i=1}^{N} \partial_{\partial D}\omega_i\,,$$

and which play the role of "trace" of the partition $(\Omega_1, \dots, \Omega_N)$. Furthermore, we prove that

- $\mathcal{F}_{\partial D}$ can be decomposed into the disjoint union of a regular part $\mathcal{R}_{\partial D}$ and a singular part $\mathcal{S}_{\partial D}$;
- $\mathcal{R}_{\partial D}$ is locally a $(d-2)$-dimensional manifold of class C^1, the modulus of continuity of the normal derivative being given in terms of the one of ∂D;
- in a neighborhood of any point in $\mathcal{R}_{\partial D}$, the interior free boundary $\mathcal{F}$ is a $(d-1)$-dimensional smooth manifold that approaches ∂D orthogonally and is the interface between two components only (like in the picture on the right in the figure below).

In particular, in view of our results, we are able to exclude pathological boundary behavior such as the ones on the left and in the center of the figure below.



The analysis of the domain walls up to ∂D faces issues, of both technical and topological nature, which are not present inside D: first, the monotonicity of the frequency function is a major technical obstacle; second, even the definition of the optimal partition on the boundary is not straightforward and requires some fine analysis of the behavior of the eigenfunctions near ∂D; third, the smoothness of the "regular part" of the free boundary cannot be deduced from the implicit function theorem, but relies on decay rate of the blow-up sequences, which we obtain through epiperimetric inequalities.

Bibliography.

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