Book of abstracts

Regularity theory for free boundary and geometric variational problems IV

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PDE analysis on stable minimal hypersurfaces: curvature estimates and sheeting

Costante Bellettini

We consider properly immersed two-sided stable minimal hypersurfaces of dimension n. We illustrate the validity of curvature estimates for $n \leq 6$ (and associated Bernstein-type properties). For $n \geq 7$ we illustrate sheeting results around "flat points". The proof relies on PDE analysis. The results extend respectively Schoen-Simon-Yau's estimates (obtained for $n \leq 5$) and Schoen-Simon's sheeting theorem (valid for embeddings).

Classical solutions for a soap film capillarity problem

Giulia Bevilacqua

In this talk, we study regularity of the soap film capillarity problem, that is soap films are chosen to be sets of finite perimeter containing an assigned amount of volume and satisfying a topological spanning condition. For plane boundaries, we show that these minimizers are normal smooth graphs with positive constant mean curvature and meeting orthogonally the boundary. This is a joint work with S. Stuvard and B. Velichkov.

Quantitative stability results via the method of moving planes

Giulio Ciraolo

The method of the moving planes is a classical tool used to prove symmetry properties for overdetermined PDE's boundary value problems and for rigidity problems in geometric analysis. In this talk we give an overview of some recent results related to quantitative studies of the method of moving planes, where quantitative approximate symmetry results are obtained.

An elementary lemma in geometric measure theory

Camillo De Lellis

Motivated by the works of Simon and Naber and Valtorta on the rectifiability of the singular set of harmonic maps and area-minimizing integral currents in codimension 1, we consider the following general question:

if a subset E of the n-dimensional Euclideal space is not $\mathcal{H}^k$-σ-finite, does it contain a purely unrectifiable subset F with $0 < \mathcal{H}^k(F) < \infty$?

We show that the answer is always affirmative under very general assumptions on the set E. This provides a shortcut for some complicated covering arguments in the work of Naber and Valtorta, and in general we believe might be useful in several other contexts. Our work, which is joint with Ian Fleschler, generalizes a classical paper of Besicovitch in the fifties, which addressed the case k=1 and n=2.

Surfaces with prescribed Gaussian images and applications to anisotropic minimal surfaces

Antonio De Rosa

We construct d-dimensional polyhedral chains in $\mathbb{R}^n$ filling the boundary of a unit d-cube and with distribution of tangent planes that is arbitrarily close to a prescribed measure on the Grassmannian having as barycenter a simple d-vector. If the measure on the Grassmannian is supported on the set of positively oriented d-planes, we can achieve these fillings as Lipschitz multigraphs. We apply this construction to prove that, for anisotropic integrands, polyconvexity is equivalent to quasiconvexity of the associated Q-integrands, and to show that strict polyconvexity is necessary for the atomic condition to hold.
Joint work with Y. Lei and R. Young.

A strict maximum principle for nonlocal minimal surfaces

Serena Dipierro

Suppose that two nonlocal minimal surfaces are included one into the other and touch at a point. Then, they must coincide. But this is perhaps less obvious than what it seems at first glance...

Regularity results for minimal capillary hypersurfaces

Nick Edelen

We describe two recent results concering the regularity of minimal capillary hypersurfaces, i.e. surfaces meeting a container at prescribed angles. The first, joint with Otis Chodosh and Chao Li, is an improved bound on the singular set for capillary minimizing hypersurfaces. We show the singular set has codimension at least 4, and this estimate improves for capillary angles close to 0, $\pi/2$, or π. The second result, joint with Luigi de Masi, Carlo Gasparetto, and Chao Li, is an Allard-type regularity theorem for capillary surfaces which are merely stationary (or have bounded mean curvature). Using the notion of capillary varifold first introduced by Kagaya-Tonegawa, we prove a sharp first-variation bound for all angles, and prove that whenever the capillary varifold is close to a capillary half-plane of angle $\neq \pi/2$, then it coincides with a $C^{1,\alpha}$ hypersurface nearby. We use our result to prove regularity at generic boundary points of density < 1.

On the one-phase problem

Xavier Fernández-Real

In this talk, we will introduce the one-phase or Alt-Caffarelli problem and present some recent advances.

Topological control for min-max free boundary minimal surfaces

Giada Franz

A free boundary minimal surface (FBMS) in a three-dimensional Riemannian manifold is a critical point of the area functional with respect to variations that constrain its boundary to the boundary of the ambient manifold. In this talk, we will focus on one of the methods that have been employed so far to construct FBMS, that is Simon-Smith variant of Almgren-Pitts min-max theory. We will see how this method allows us to control the topology (i.e. genus and number of boundary components) of the resulting surface, and we will present several applications.

On the Lawson-Osserman conjecture on the minimal surfaces system

Jonas Hirsch

In the renowned paper by Lawson and Osserman, non-existence, non-uniqueness, and irregularity of solutions to the minimal surface system, Conjecture 2.1 stands out:

Conjecture 2.1: The systems (2.2) and (2.3) are equivalent for any locally Lipschitz function f on Ω.

Here (2.2) is the full minimal surface system

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij}\right) = 0; \qquad j = 1, \dots, n; \\ \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x_j}\right) = 0. \end{cases}$$

We affirmatively resolve the conjecture in dimension two. Our main result can be succinctly stated as follows:

Theorem: Let $f : B_1 \subset \mathbb{R}^2 \to \mathbb{R}^n$ be a Lipschitz critical point of the area functional concerning outer variations. Then f is smooth.

Having presented the conjecture and our result, the remainder of the talk will be devoted to outlining the ideas behind the proof and elucidating the role of working in two dimensions.

This is a joint work with Connor Mooney and Riccardo Tione.

Stationary solutions to the Bernoulli free boundary problem

Dennis Kriventsov

Free boundary problems of Bernoulli type arise naturally in fluid dynamics, thermal models, shape optimization, and other contexts. We will focus on the simplest possible archetype problem, and consider many examples of solutions in one and two dimensions. Then we will look at the question of which kinds of solutions are closed under taking limits, and how those limits look like?this is a topic of practical importance for constructing solutions by any argument save for direct minimization. Thanks to a recent breakthrough in joint work with Georg Weiss, we can now give a very precise and descriptive answer to such questions.

Recent developments on Besicovitch's 1/2 problem

Annalisa Massaccesi

Besicovitch's $\frac{1}{2}$ problem investigates the smallest threshold σ guaranteeing rectifiability for a set with Hausdorff 1-dimensional finite measure when the lower density of the set is larger than σ almost everywhere. Besicovitch conjectured that $\sigma = \frac{1}{2}$ (hence the name of the problem) and proved $\sigma \leq \frac{3}{4}$, then Preiss and Tiser improved the bound to

$$\sigma \leq \frac{2+\sqrt{46}}{12} \sim 0.73186....$$

In a recent work in collaboration with C. De Lellis, F. Glaudo and D. Vittone, we devise a strategy to improve the bound by means of a hierarchy of variational problems and we reach a proof that $\sigma \leq 0.7$. In this seminar, I will try to explain the fairly intuitive geometric idea behind this strategy and I will try to summarize both the computational obstacles and the intrinsic obstacles that are still in the way.

Uniqueness of regular tangent cones for immersed stable minimal hypersurfaces

Paul Minter

Whether tangent cones are unique is a natural question to ask when one wants to understand singularities in minimal surfaces. One key obstacle however is if the tangent cone occurs with multiplicity. I will discuss joint work with Nick Edelen in which we are able to prove such uniqueness (with a decay estimate and local multi-valued graph structure) at certain tangent cones for immersed stable minimal hypersurfaces (in particular those with a smooth link, allowing for multiplicity). Similar results also hold for tangent cones at infinity. We also construct examples showing the need for multi-valued functions in this problem. Our proof utilises ideas of Leon Simon, namely the Łojasiewicz-Simon inequality.

Regularity of domain walls in optimal partition problems

Roberto Ognibene

Let us consider a bounded domain, divided into a fixed number of disjoint subdomains and, among all the possible configurations, let us consider the one for which the sum of the first Dirichlet eigenvalues of the subdomains is minimal. In this talk, I will discuss the regularity of the interface which emerges as boundary of such optimal partition and, in particular, I will focus on the regularity up to the fixed boundary. The talk is based on a joint work with B. Velichkov.

Boundary behavior of the Allen-Cahn equation: free boundary integral varifolds, and stable curvature estimates

Davide Parise

In the late 70s, the work of Modica, Mortola, De Giorgi, and many others established deep connections between the Allen-Cahn equation, a semi-linear elliptic equation arising in the van der Waals-Cahn-Hilliard theory of phase transitions, and minimal hypersurfaces, i.e. critical points of the area functional. Based on these ideas, in recent years, the combined work of Guaraco, Hutchinson, Tonegawa, and Wickramasekera established the existence of (optimally regular) minimal hypersurfaces in compact manifolds without boundary. In this talk we will consider the Allen-Cahn equation on manifolds with boundary, and describe geometric and analytic aspects of the boundary behavior of the associated limit interfaces. The end goal of this line of investigation is the construction of free boundary minimal hypersurfaces in manifolds with boundary, i.e. submanifolds with vanishing mean curvature and meeting the boundary orthogonally. I will present progress towards this goal, based on joint works with Wenkui Du, Akashdeep Day, Martin Li, and Lorenzo Sarnataro.

Remarks on stable surfaces in higher codimension

Richard Schoen

This talk will give an informal discussion of stable surfaces in high dimensional euclidean spaces or Riemannian manifolds. We will discuss results which are known and results which might be expected to hold.

Mean Curvature Flow from conical singularities

Felix Schulze

We give a proof of Ilmanen's resolution of point singularities conjecture by establishing short-time smoothness of the level set flow of a smooth hypersurface with isolated conical singularities. Combined with the uniqueness of asymptotically conical tangent flows, this shows how the outermost mean curvature flows evolve through such singularities and how mean curvature flow becomes non-unique past such singularities. Furthermore, we resolve a particular case of Ilmanen's strict genus reduction conjecture. Precisely, we prove that the level set flow of a smooth hypersurface $M^n \subset \mathbb{R}^{n+1}$, $2 \leq n \leq 6$, with an isolated conical singularity is modelled on the level set flow of the cone. In particular, the flow fattens (instantaneously) if and only if the level set flow of the cone fattens. This is joint work with Otis Chodosh and Joshua Daniels-Holgate.

Singularly perturbed elliptic systems modeling partial separation and their free boundaries

Nicola Soave

Various physical phenomena can be described by a certain number of densities (mass density, population density, probability density) whose distribution is governed by principles of diffusion and mutual interaction. In this seminar, we present some recent results concerning an elliptic problem in which the interaction between the different components is partially competitive, in an appropriate sense. We will discuss results regarding uniform bounds for the problem with singular perturbation, optimal regularity of the limit problem, and regularity of the free boundary. We will then present a number of open problems. This is an ongoing project with Susanna Terracini.

Brakke flow with a forcing term

Yoshihiro Tonegawa

Given a compact smooth hypersurface and a non-smooth time-dependent vector field, one can prove some existence of evolving hypersurface whose velocity is equal to the mean curvature and the given vector field. I discuss the existence and regularity of such hypersurface under a certain general regularity assumption on the vector field which is subcritical in the sense of parabolic scaling. I also present a recent existence result which may be considered a critical case.

Sheet happens (but only as the root of 1-s)

Enrico Valdinoci

We discuss the regularity properties of two-dimensional stable s-minimal surfaces, presenting a robust $C^{2,\alpha}$-estimate and an optimal sheet separation bound, according to which the distance between different connected components of the surface must be at least the square root of 1-s.

Energy identity for stationary harmonic maps

Daniele Valtorta

We present the proof for Energy Identity for stationary harmonic maps. In particular, given a sequence of stationary harmonic maps weakly converging to a limit with a defect measure for the energy, then m-2 almost everywhere on the support of this measure the density is the sum of energy of bubbles. This is equivalent to saying that annular regions (or neck regions) do not contribute to the energy of the limit.

This result is obtained via a quantitative analysis of the energy in annular regions for a fixed stationary harmonic map. The proof is technically involved, but it will be presented in simplified cases to try and convey the main ideas behind it.